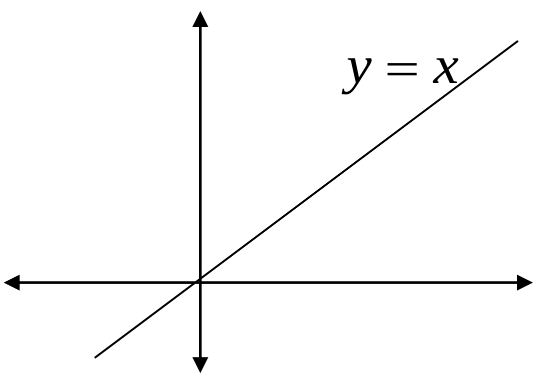
# 8. FUNGSI TRANSENDEN

# 8.1 Fungsi Invers

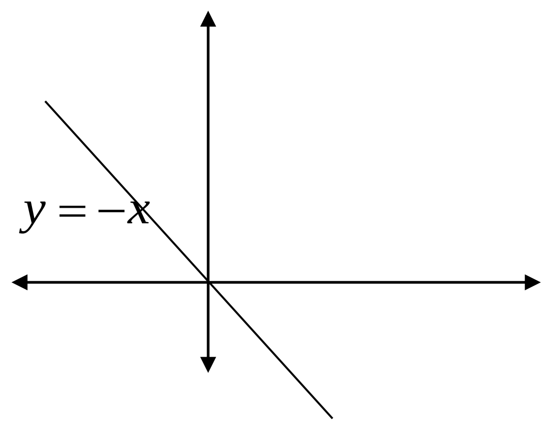
Misalkan $f : D_f \to R_f$ dengan

$$x \mapsto y = f(x)$$

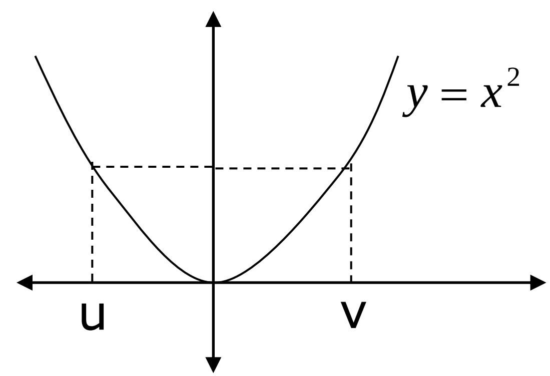
**Definisi 8.1** Fungsi $y = f(x)$ disebut satu-satu jika $f(u) = f(v)$ maka $u = v$ atau jika $u \neq v$ maka $f(u) \neq f(v)$




fungsi y=x satu-satu




fungsi y=-x satu-satu

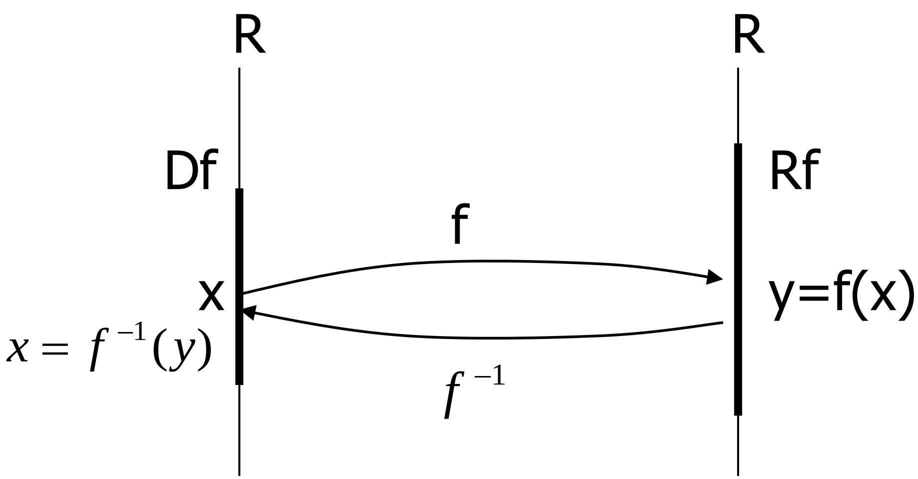



fungsi $y = x^2$ tidak satu-satu

Secara geometri grafik fungsi satu-satu dan garis yang sejajar dengan sumbu x berpotongan di satu titik.

Teorema : Jika fungsi f satu-satu maka f mempunyai invers notasi $f^{-1}$

$$f^{-1} : R_f \to D_f$$
$$y \mapsto x = f^{-1}(y)$$




Berlaku hubungan

$$f^{-1}(f(x)) = x$$

$$f(f^{-1}(y)) = y$$

$$D_{f^{-1}} = R_f, \quad R_{f^{-1}} = D_f$$

Teorema : jika f monoton murni(selalu naik/selalu turun) maka
f mempunyai invers

f(x)=x

$f'(x) = 1 > 0, \forall x \in R$

f selalu naik

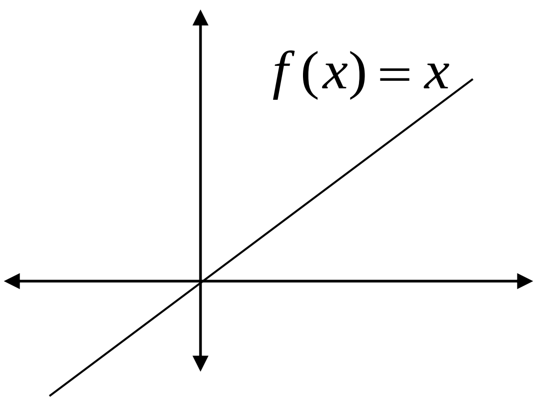
f(x)=-x

$f'(x) = -1 < 0, \forall x \in R$

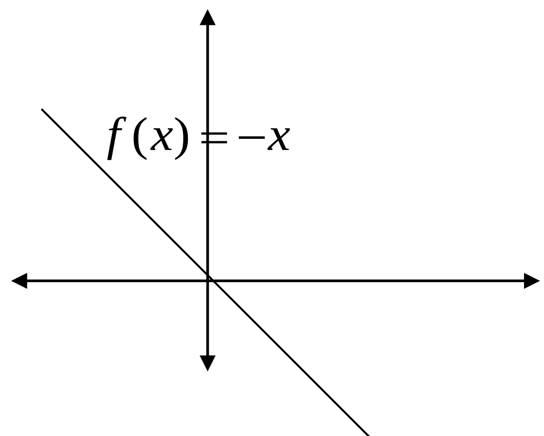
f selalu turun

$$f'(x) = 2x = \begin{cases} > 0, x > 0 \\ < 0, x < 0 \end{cases}$$

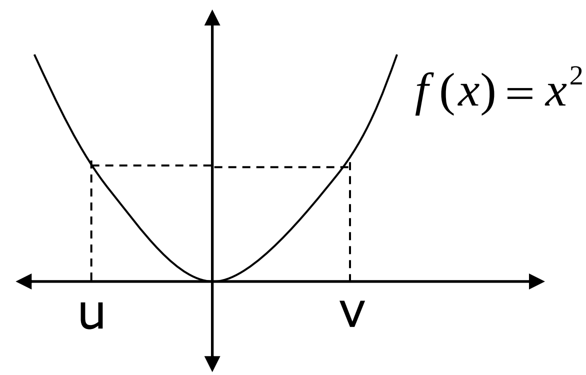
f naik untuk x>0
turun untuk x <0




$f^{-1}$ ada




$f^{-1}$ ada




$f^{-1}$ tidak ada

Contoh : Diketahui $f(x) = \frac{x-1}{x+2}$

a. Periksa apakah f mempunyai invers

b. Jika ada, tentukan inversnya

Jawab

a. $f'(x) = \frac{1.(x+2) - 1.(x-1)}{(x+2)^2} = \frac{3}{(x+2)^2} > 0, \forall x \in Df$

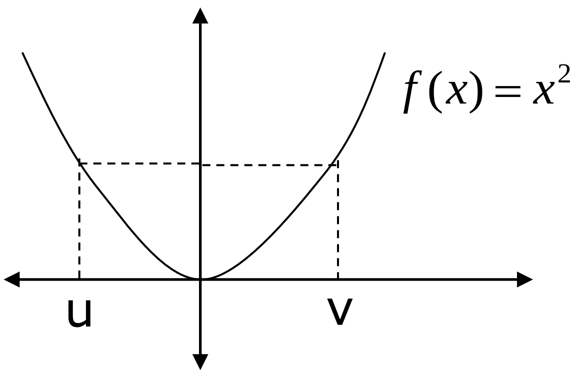
Karena f selalu naik(monoton murni) maka f mempunyai invers

b. Misal $y = \frac{x-1}{x+2}$

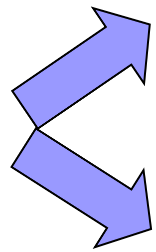
$$xy + 2y = x - 1 \Leftrightarrow xy - x = -2y - 1 \Rightarrow x = \frac{-2y-1}{y-1}$$

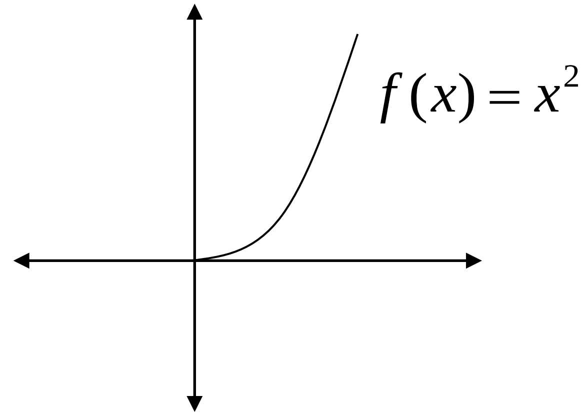
$$f^{-1}(y) = \frac{-2y-1}{y-1} \Rightarrow f^{-1}(x) = \frac{-2x-1}{x-1}$$

Suatu fungsi yang tidak mempunyai invers pada daerah asalnya dapat dibuat mempunyai invers dengan cara membatasi daerah asalnya.

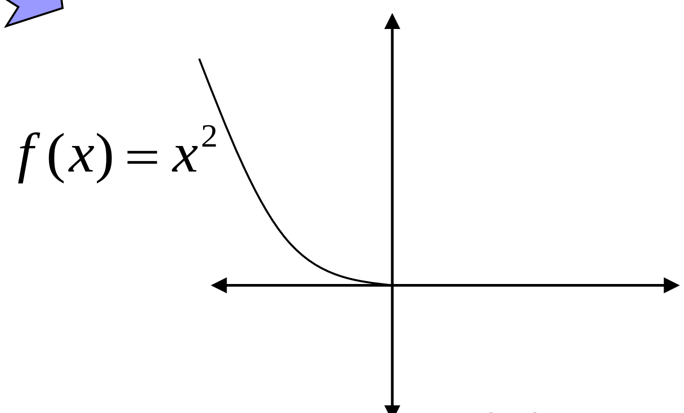



Untuk $x \in R$ $f^{-1}$ tidak ada



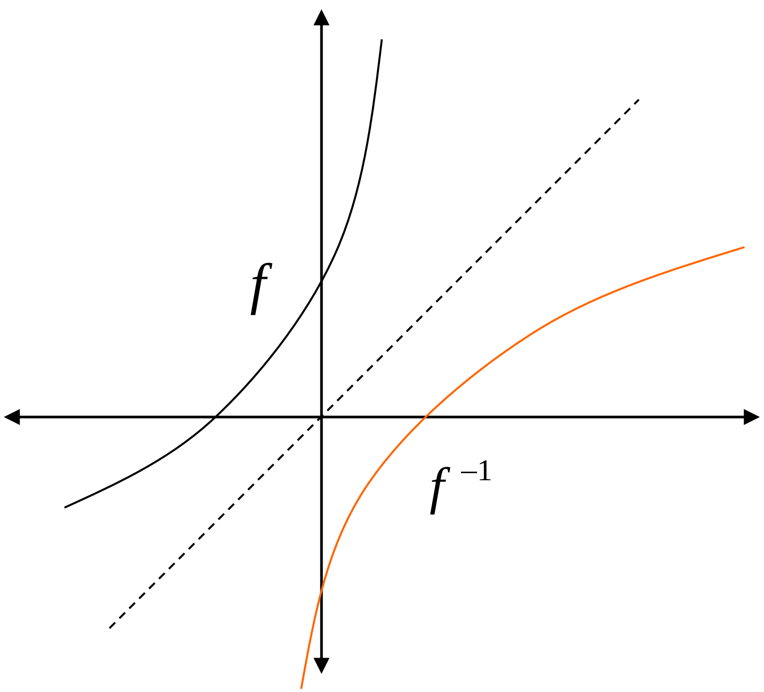



Untuk x>0 $f^{-1}$ ada




Untuk x<0 $f^{-1}$ ada

# Grafik fungsi invers




Titik (x,y) terletak pada grafik f $\Longleftrightarrow$ Titik (y,x) terletak pada grafik $f^{-1}$

Titik (x,y) dan (y,x) simetri terhadap garis y=x

Grafik f dan $f^{-1}$ semetri terhadap garis y=x

## Turunan fungsi invers

**Teorema** Misalkan fungsi f monoton murni dan mempunyai turunan pada selang I. Jika $f^{-1}(x) \neq 0, x \in I$ maka $f^{-1}$ dapat diturunkan di y=f(x) dan

$$(f^{-1})'(y) = \frac{1}{f(x)}$$

Bentuk diatas dapat juga dituliskan sebagai

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Contoh Diketahui $f(x) = x^5 + 2x + 1$ tentukan $(f^{-1})'(4)$

Jawab :

$f'(x) = 5x^4 + 2$ ,y=4 jika hanya jika x=1

$$\Rightarrow (f^{-1})'(4) = \frac{1}{f'(1)} = \frac{1}{7}$$

## Soal Latihan

Tentukan fungsi invers ( bila ada ) dari

1. $f(x) = x + \dfrac{1}{x} \quad , \quad x > 0$
2. $f(x) = \sqrt[3]{2x-1}$
3. $f(x) = \sqrt[5]{4x+2}$
4. $f(x) = \dfrac{5}{x^2+1} \quad , \quad x \geq 0$
5. $f(x) = \dfrac{x+1}{x-1}$
6. $f(x) = \dfrac{2x-3}{x+2}$

# 8.2 Fungsi Logaritma Asli

- Fungsi Logaritma asli ( ln ) didefinisikan sebagai :

$$\ln x = \int_1^x \frac{1}{t}\,dt,\ x > 0$$

- Dengan Teorema Dasar Kalkulus II, diperoleh :

.

$$D_x[\ln x] = D_x\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

- Secara umum, jika $u = u(x)$ maka

$$D_x[\ln u] = D_x\left(\int_1^{u(x)} \frac{1}{t}\,dt\right) = \frac{1}{u}\frac{du}{dx}$$

Contoh : Diberikan $f(x) = \ln(\sin(4x+2))$

maka $f'(x) = \dfrac{1}{\sin(4x+2)} D_x(\sin(4x+2)) = 4\cot(4x+2)$

Jika $y = \ln|x|, x \neq 0$

$$= \begin{cases} \ln x, \; x > 0 \\ \ln(-x), \; x < 0 \end{cases}$$

$\ln x, x>0 \rightarrow y = \ln x \rightarrow y' = \dfrac{1}{x}$

$\ln(-x), x<0 \rightarrow y = \ln(-x) \rightarrow y' = \dfrac{-1}{-x} = \dfrac{1}{x}$

Jadi, $\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x} \quad , x \neq 0.$

Dari sini diperoleh : $\int \dfrac{1}{\mathrm{x}} \mathrm{dx} = \ln|\mathrm{x}| + \mathrm{C}$

## Sifat-sifat Ln :

1. ln 1 = 0
2. ln(*ab*) = ln *a* + ln *b*
3. ln(*a*/*b*)=ln(*a*) – ln(*b*)
4. $\ln a^r = r \ln a$

Contoh: Hitung $\int_0^4 \frac{x^2}{x^3+2}dx$

jawab

Misal $u = x^3 + 2 \rightarrow du = 3x^2 dx$

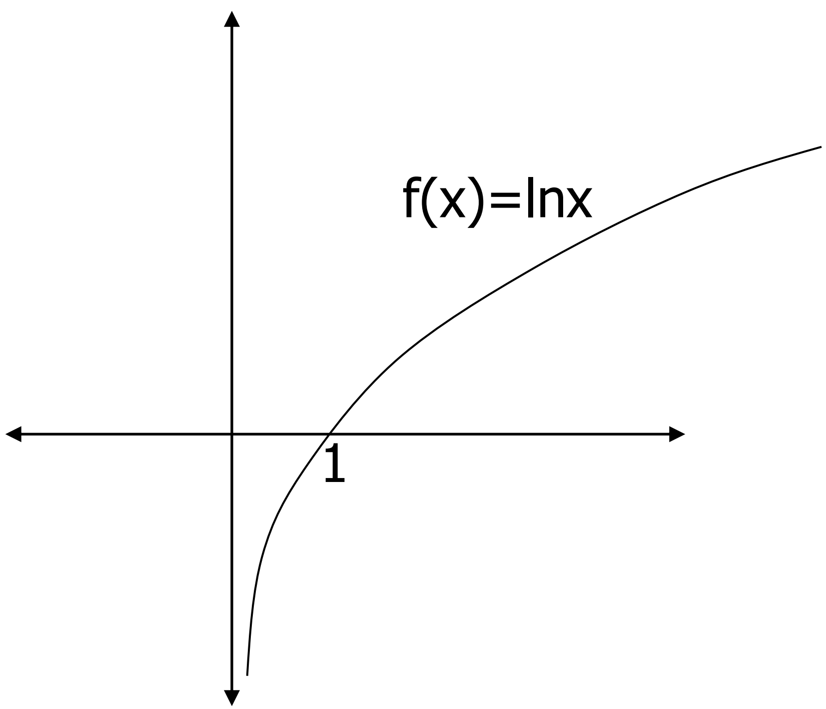
$$\int \frac{x^2}{x^3+2}dx = \int \frac{x^2}{u}\frac{du}{3x^2} = \frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + c$$

$$= \frac{1}{3}\ln|x^3+2| + c$$

sehingga

$$\int_0^4 \frac{x^2}{x^3+2}dx = \frac{1}{3}\ln|x^3+2|\Big]_0^4 = \frac{1}{3}(\ln 66 - \ln 2) = \frac{1}{3}\ln 33.$$

# Grafik fungsi logaritma asli




Diketahui

a. $f(x) = \ln x = \int_1^x \frac{dt}{t}, x > 0$

b. $f'(x) = \frac{1}{x} > 0 \, \forall x \in D_f$

f selalu monoton naik pada Df

c. $f''(x) = -\frac{1}{x^2} < 0 \, \forall x \in D_f$

Grafik selalu cekung kebawah

d. f(1) = 0

## 8.3 Fungsi Eksponen Asli

- Karena $D_x[\ln x] = \frac{1}{x} > 0 \text{ untuk } x > 0,$ maka fungsi logaritma asli monoton murni, sehingga mempunyai invers. Invers dari fungsi logaritma asli disebut **fungsi eksponen asli,** notasi **exp**. Jadi berlaku hubungan

$$y = \exp(x) \Leftrightarrow x = \ln y$$

- Dari sini didapat : y = exp(ln y) dan x =ln(exp(x))
- **Definisi 8.2** Bilangan *e* adalah bilangan Real positif yang bersifat ln *e* = 1.

  Dari sifat (iv) fungsi logaritma diperoleh

$$e^r = \exp(\ln e^r) = \exp r \ln e = \exp r \quad \Rightarrow \quad \exp(x) = e^x$$

## Turunan dan integral fungsi eksponen asli

Dengan menggunakan turunan fungsi invers

Dari hubungan

$$y = e^x \quad \Leftrightarrow \quad x = \ln y$$

$$\frac{dy}{dx} = \frac{1}{dx/dy} = y = e^x \quad \Leftarrow \quad \frac{dx}{dy} = \frac{1}{y}$$

Jadi, $D_x(e^x) = e^x$

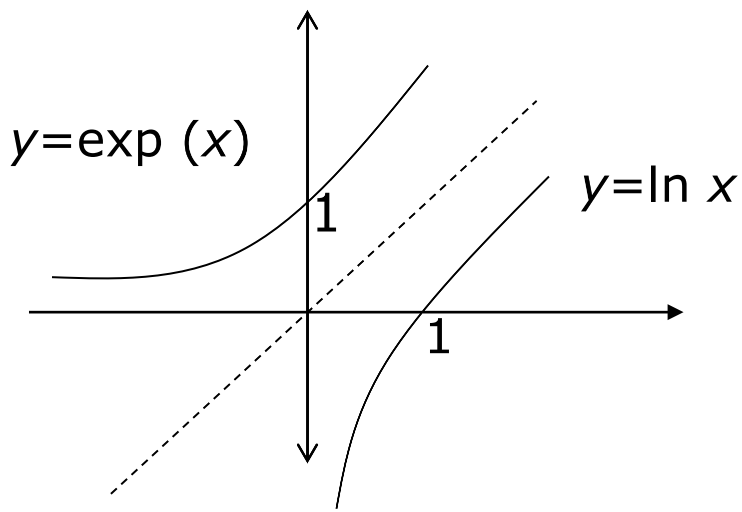
Secara umum $D_x(e^{u(x)}) = e^u.u'$

Sehingga

$$\int e^x\,dx = e^x + C$$

## Grafik fungsi eksponen asli

Karena fungsi ekponen asli merupakan invers dari fungsi logaritma asli maka grafik fungsi eksponen asli diperoleh dengan cara mencerminkan grafik fungsi logaritma asli terhadap garis y=x




Contoh

$$D_x(e^{3x\ln x}) = e^{3x\ln x}.D_x(3x\ln x) \quad = e^{3x\ln x}(3\ln x + 3).$$

Contoh Hitung

$$\int \frac{e^{3/x}}{x^2} dx$$

Jawab :

Misalkan $u = \frac{3}{x} \rightarrow du = \frac{-3}{x^2} dx \rightarrow \frac{1}{x^2} dx = -\frac{1}{3} du$

Sehingga

$$\int \frac{e^{3/x}}{x^2} dx = \int -\frac{1}{3} e^u du = -\frac{1}{3} e^u + c = -\frac{1}{3} e^{3/x} + c.$$

# Penggunaan fungsi logaritma dan eksponen asli

a. Menghitung turunan fungsi berpangkat fungsi

Diketahui $f(x) = (g(x))^{h(x)}$ , $f'(x) = ?$

$$\ln(f(x)) = h(x)\ln(g(x))$$

$$D_x(\ln(f(x))) = D_x(h(x)\ln(g(x)))$$

$$\frac{f'(x)}{f(x)} = h'(x)\ \ln(g(x)) + \frac{h(x)}{g(x)} g'(x)$$

$$f'(x) = \left( h'(x)\ \ln(g(x)) + \frac{h(x)}{g(x)} g'(x) \right) \ f(x)$$

## Contoh

Tentukan turunan fungsi $f(x) = (\sin x)^{4x}$

Jawab

Ubah bentuk fungsi pangkat fungsi menjadi perkalian fungsi dengan menggunakan fungsi logaritma asli

$$\ln f(x) = \ln(\sin x)^{4x} = 4x \ln(\sin(x))$$

Turunkan kedua ruas

$$D_x(\ln f(x)) = D_x(4x \ln(\sin(x)))$$

$$\frac{f'(x)}{f(x)} = 4\ln(\sin(x)) + \frac{4x}{\sin x}\cos x = 4\ln(\sin(x)) + 4x\cot x$$

$$f'(x) = (4\ln(\sin(x)) + 4x\cot x)(\sin x)^{4x}$$

## b. Menghitung limit fungsi berpangkat fungsi

$$\lim_{x \to a} f(x)^{g(x)} = ?$$

Untuk kasus

(i) $\lim_{x \to a} f(x) = 0, \quad \lim_{x \to a} g(x) = 0$

(ii) $\lim_{x \to a} f(x) = \infty, \quad \lim_{x \to a} g(x) = 0$

(iii) $\lim_{x \to a} f(x) = 1, \quad \lim_{x \to a} g(x) = \infty$

Penyelesaian :

Tulis $\lim_{x \to a} ( f(x)^{g(x)} ) = \lim_{x \to a} [\exp (\ln f(x)^{g(x)} )] = \lim_{x \to a} \exp g(x)[\ln f(x)]$

Karena fungsi eksponen kontinu, maka

$$\lim_{x \to a} \exp g(x) \ln ( f(x) = \exp \lim_{x \to a} g(x) \ln f(x)$$

Contoh Hitung

a. $\lim_{x\to 0^+} x^x$  b. $\lim_{x\to 0} (1+x)^{\frac{1}{x}}$  c. $\lim_{x\to\infty} \left(x^3+1\right)^{1/\ln x}$

Jawab

a. $\lim_{x\to 0^+} (x^x) = \exp \lim_{x\to 0^+} x \ln x$ (bentuk $0.\infty$ )

Rubah ke bentuk $\infty/\infty$ lalu gunakan dalil L'hopital

$$= \exp \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \exp \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \exp(0) = 1$$

b. $\lim_{x\to 0} ((1+x)^{1/x}) = \lim_{x\to 0} \exp .\frac{1}{x}.\ln(1+x) = \exp \lim_{x\to 0} \frac{\ln(1+x)}{x}$

Gunakan dalil L'hopital

$$= \exp \lim_{x\to 0} \frac{1}{1+x} = \exp 1 = e$$

sehingga

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

c. $\lim_{x\to\infty} \left(x^3+1\right)^{1/\ln x} = \exp \lim_{x\to\infty} \frac{1}{\ln x} \ln(x^3+1) \quad = \exp \lim_{x\to\infty} \frac{\ln(x^3+1)}{\ln x}$

Gunakan dalil L'hopital

$$= \exp \lim_{x\to\infty} \frac{\frac{3x^2}{x^3+1}}{1/x}. \quad = \exp \lim_{x\to\infty} \frac{3x^3}{x^3+1} \quad = \exp \lim_{x\to\infty} \frac{x^3(3)}{x^3(1+\frac{1}{x^3})}$$

$$= \exp \lim_{x\to\infty} \frac{(3)}{(1+\frac{1}{x^3})} \qquad = \exp 3 = e^3.$$

## Soal latihan

A. Tentukan $y'$ dari

1. $y = \sec e^{2x} + e^{2\sec x}$
2. $y = x^5 e^{-3\ln x}$
3. $y = \tan e^{\sqrt{x}}$
4. $y^2 e^{2x} + xy^3 = 1$
5. $e^y = \ln(x^3 + 3y)$
6. $y = \ln\left(x^2 - 5x + 6\right)$
7. $y = \ln \cos 3x$
8. $y = \dfrac{\ln x}{x^2}$
9. $y = \ln\left(\sin x\right)$
10. $y = \sin(\ln(2x+1))$

## B. Selesaikan integral tak tentu berikut

1. $\int \frac{4}{2x+1}\,dx$
2. $\int \frac{\ln^2 3x}{x}\,dx$
3. $\int \frac{x^3}{x^2+1}\,dx$
4. $\int \frac{\tan(\ln x)}{x}\,dx$
5. $\int \frac{2}{x(\ln x)^2}\,dx$
6. $\int \frac{4x+2}{x^2+x+5}\,dx$
7. $\int (x+3)\,e^{x^2+6x}\,dx$
8. $\int e^{-x}\sec^2\left(2-e^{-x}\right)dx$
9. $\int (\cos x)\,e^{\sin x}\,dx$
10. $\int e^{2\ln x}\,dx$
11. $\int x^2 e^{2x^3}\,dx$
12. $\int \frac{e^{2x}}{e^x+3}\,dx$
13. $\int \frac{e^{3x}}{(1-2e^{3x})^2}\,dx$

## C. Selesaikan integral tentu berikut

1. $\displaystyle\int_1^4 \frac{3}{1-2x}\,dx$

2. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)}\,dx$

3. $\displaystyle\int_{-\ln 3}^{\ln 3} \frac{e^x}{e^x+4}\,dx$

4. $\displaystyle\int_0^{\ln 5} e^x\left(3-4e^x\right)\,dx$

5. $\displaystyle\int_0^1 e^{2x+3}dx$

6. $\displaystyle\int_0^{\ln 2} e^{-3x}dx$

7. $\displaystyle\int_1^2 \frac{e^{3/x}}{x^2}\,dx$

8. $\displaystyle\int_0^2 xe^{4-x^2}\,dx$

9. $\displaystyle\int_e^{e^2} \frac{dx}{x(\ln x)^2}$

D. Hitung limit berikut :

1. $\lim_{x \to 1} x^{\frac{1}{1-x}}$

2. $\lim_{x \to 0} (1 + \sin 2x)^{\frac{1}{x}}$

3. $\lim_{x \to \infty} \left[ \cos \frac{2}{x} \right]^{x^2}$

4. $\lim_{x \to 0^+} \left( e^{2x} - 1 \right)^{\frac{1}{\ln x}}$

5. $\lim_{x \to \infty} \left( 1 + x^2 \right)^{\frac{1}{\ln x}}$

6. $\lim_{x \to \infty} (\ln x)^{\frac{1}{x}}$

7. $\lim_{x \to \infty} \left( 3^x + 5^x \right)^{\frac{1}{x}}$

8. $\lim_{x \to \infty} \left( \frac{x+1}{x+2} \right)^x$

# 8.5 Fungsi Eksponen Umum

Fungsi $f(x) = a^x$, *a* > 0 disebut fungsi eksponen umum

Untuk *a* > 0 dan *x* ∈ ***R***, didefinisikan $a^x = e^{x \ln a}$

Turunan dan integral

$$D_x(a^x) = D_x(e^{x\ln a}) = e^{x \ln a} \ln a = a^x \ \ln a$$

Jika *u* = *u*(*x*), maka

$$D_x(a^u) = D_x(e^{u\ln a}) = e^{u \ln a} \ln a . u' = a^u \ u' \ln a$$

Dari sini diperoleh :

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

## Sifat–sifat fungsi eksponen umum

Untuk $a > 0$, $b > 0$, $x$, $y$ bilangan riil berlaku

1. $a^x a^y = a^{x+y}$

2. $\dfrac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = a^{xy}$

4. $(ab)^x = a^x b^x$

5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Contoh

1. Hitung turunan pertama dari

$$f(x) = 3^{2x+1} + 2^{\sin 2x}$$

Jawab :

$$f'(x) = 2.3^{2x+1} \ln 3 + 2.2^{\sin 2x} \ln 2$$

2. Hitung

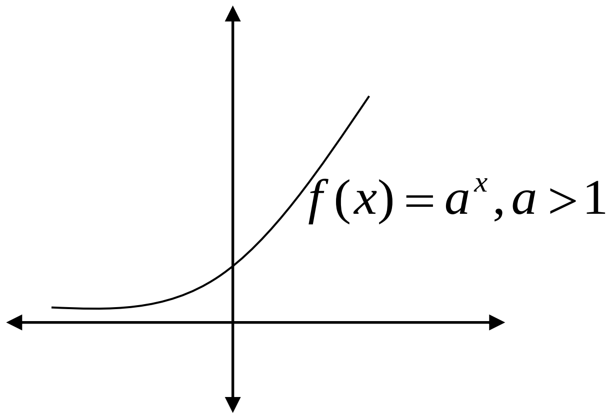
$$\int 4^{x^2} . x dx$$

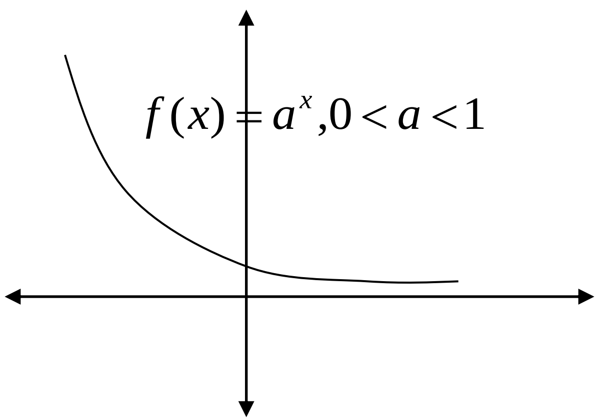
Jawab :

Misal $u = x^2 \rightarrow du = 2x dx \rightarrow dx = \frac{1}{2x} du$

$$\Rightarrow \int 4^{x^2} . x dx = \int 4^u \frac{du}{2} = \frac{1}{2} \frac{4^u}{\ln 4} + C = \frac{4^{x^2}}{2 \ln 4} + C$$

# Grafik fungsi eksponen umum




Diketahui

a. $f(x)=a^x \;, a>0$

$Df = (-\infty, \infty)$

b. $f\,'(x)=a^x \ln a \;= \begin{cases} a^x \ln a < 0 \;,\; 0<a<1 \\ a^x \ln a > 0 \;,\; a>1 \end{cases}$

f monoton naik jika a > 1
monoton turun jika 0 < a < 1

c. $f\,''(x)=a^x (\ln a)^2 > 0 \;\forall x \in D_f$

Grafik f selalu cekung keatas

d. f(0) = 1

# 8.6 Fungsi Logaritma Umum

Karena fungsi eksponen umum monoton murni maka ada Inversnya. Invers dari fungsi eksponen umum disebut fungsi Logaritma Umum ( logaritma dengan bilangan pokok *a* ), notasi ${}^{a}\log x$ , sehingga berlaku :

$$y = {}^{a}\log x \Leftrightarrow \mathrm{x} = \mathrm{a}^{\mathrm{y}}$$

Dari hubungan ini, didapat

$$\ln\ x = \ln a^{y} = y \ln a \ \Rightarrow \ y = \frac{\ln x}{\ln a} \ \Rightarrow \ {}^{a}\log x = \frac{\ln x}{\ln a}$$

Sehingga $D_x({}^{a}\log x) = D_x(\frac{\ln x}{\ln a}) = \frac{1}{x \ln a}$

Jika *u*=*u*(*x*), maka $D_x({}^{a}\log u) = D_x(\frac{\ln u}{\ln a}) = \frac{u'}{u \ln a}$

Contoh Tentukan turunan pertama dari

1. $f(x) = {}^{3}\log(x^2 + 1)$

2. $f(x) = {}^{4}\log\left(\frac{x+1}{x-1}\right)$

Jawab :

1. $f(x) = {}^{3}\log(x^2 + 1) = \frac{\ln(x^2+1)}{\ln 3}$ ⇒ $f'(x) = \frac{2x}{x^2+1}\frac{1}{\ln 3}$

2. $f(x) = {}^{4}\log\left(\frac{x+1}{x-1}\right) = \frac{\ln\left(\frac{x+1}{x-1}\right)}{\ln 4}$ ⇒ $f'(x) = \frac{1}{\ln 4}\frac{1}{\left(\frac{x+1}{x-1}\right)} Dx\left(\frac{x+1}{x-1}\right)$

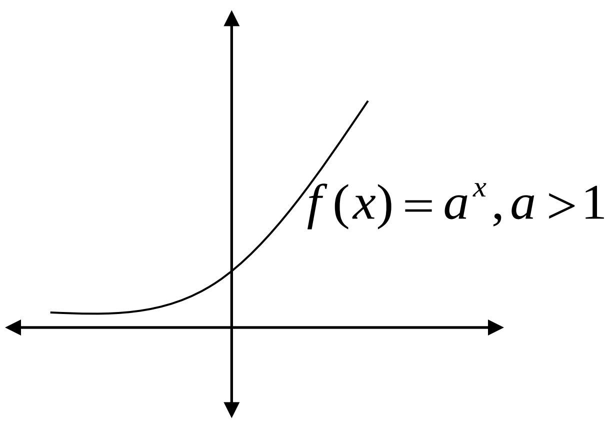
$$= \frac{1}{\ln 4}\frac{x-1}{x+1}\frac{x-1-(x+1)}{(x-1)^2}$$

$$= \frac{1}{\ln 4}\frac{-2}{(x+1)(x-1)}$$

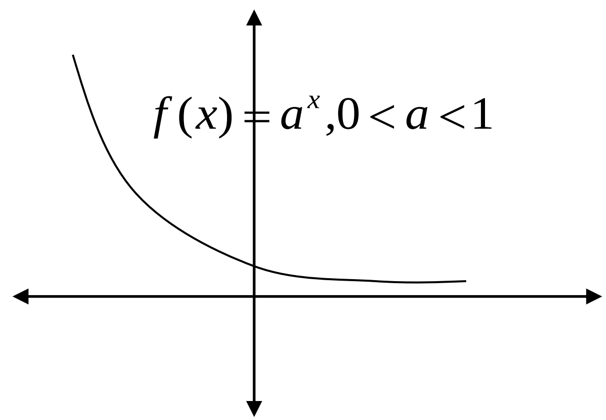
# Grafik fungsi logaritma umum

Grafik fungsi logaritma umum diperoleh dengan mencerminkan grafik fungsi eksponen umum terhadap garis y=x

Untuk a > 1




Untuk 0 < a < 1



## Soal Latihan

A. Tentukan $y'$ dari

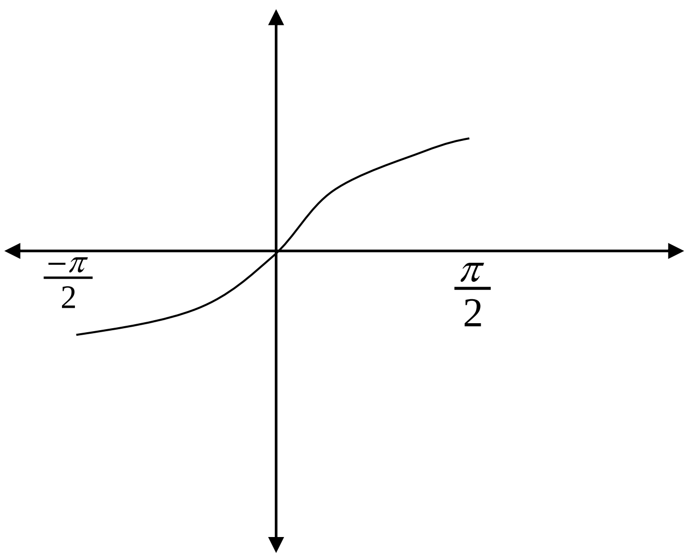
1. $y = 3^{2x^4 - 4x}$
2. $y = {}^{10}\log\left(x^2 + 9\right)$
3. $x\ {}^{3}\log(xy) + y = 2$

B. Hitung

1. $\int 10^{5x-1} dx$
2. $\int x\ 2^{x^2}\ dx$

# 8.7 Fungsi Invers Trigonometri

Fungsi trigonometri adalah fungsi yang periodik sehingga tidak satu-satu, jika daerah asalnya dibatasi fungsi trigonometri bisa dibuat menjadi satu-satu sehingga mempunyai invers.




a. Invers fungsi sinus

Diketahui f(x) = sinx , $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$

Karena pada $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$ f(x)=sinx monoton murni maka inversnya ada. Invers dari fungsi sinus disebut arcus sinus, notasi arcsin(x),atau $\sin^{-1}(x)$

Sehingga berlaku

$$y = \sin^{-1} x \Leftrightarrow x = \sin y$$

# Turunan

Dari hubungan $y = \sin^{-1} x \Leftrightarrow \quad x = \sin y \quad -1 \le x \le 1, \frac{-\pi}{2} \le y \le \frac{\pi}{2}$

dan rumus turunan fungsi invers diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}, |x|<1$$

atau $D_x(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

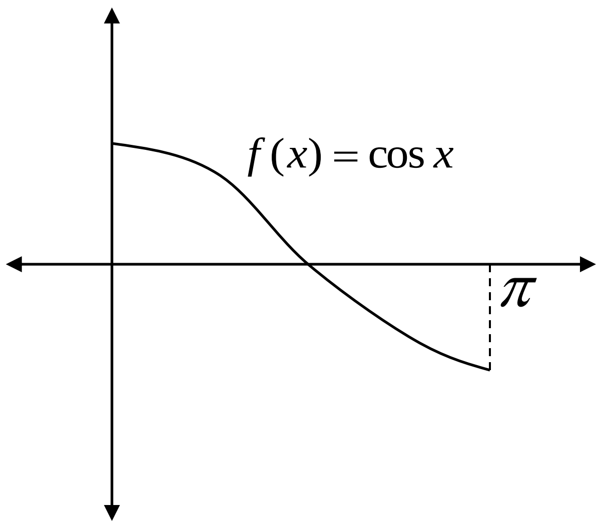
Jika u=u(x) $D_x(\sin^{-1} u) = \dfrac{u'}{\sqrt{1-u^2}}$

Dari rumus turunan diperoleh

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

## b. Invers fungsi cosinus

Fungsi f(x) = cosx $, 0 \leq x \leq \pi$ monoton murni(selalu monoton turun), sehingga mempunyai invers




**Definisi** : Invers fungsi cosx disebut arcuscosx, notasi arc cosx atau $\cos^{-1}(x)$

Berlaku hubungan

$$y = \cos^{-1} x \Leftrightarrow x = \cos y$$

**Turunan**

Dari $y = \cos^{-1} x \Leftrightarrow x = \cos y \quad , -1 \leq x \leq 1, 0 \leq y \leq \pi$ diperoleh

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}, |x| < 1$$

atau $D_x(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

Jika u= u(x) $D_x(\cos^{-1} u) = \dfrac{-u'}{\sqrt{1-u^2}}$

Dari rumus turunan diatas diperoleh

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$$

Contoh

$$D_x(\sin^{-1}(x^2)) = \frac{1}{\sqrt{1-(x^2)^2}} D_x(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

$$D_x(\cos^{-1}(\tan x)) = \frac{-1}{\sqrt{1-(\tan x)^2}} D_x(\tan x) = \frac{-\sec^2 x}{\sqrt{1-\tan^2 x}}$$

**Contoh** Hitung

$$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

Gunakan rumus

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(u) + C$$

Jawab :

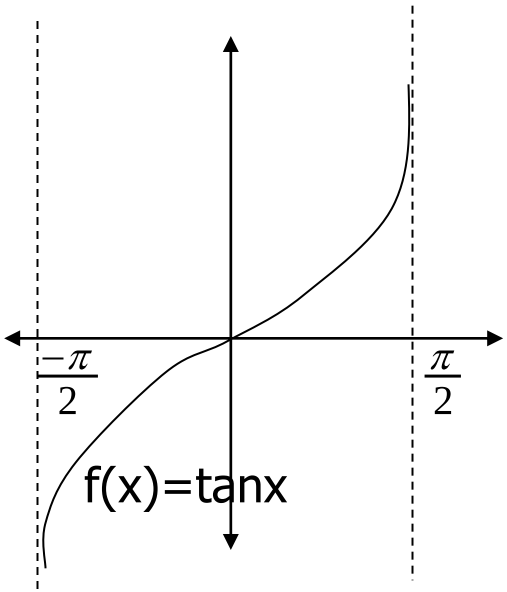
$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4(1-\frac{x^2}{4})}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{(1-(\frac{x}{2})^2}}\,dx$$

Misal $u = \dfrac{x}{2} \rightarrow du = \frac{1}{2}\,dx \rightarrow dx = 2du$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \frac{1}{2}\int \frac{2}{\sqrt{(1-u^2}}\,du = \sin^{-1} u + C = \sin^{-1}(\tfrac{x}{2}) + C$$

## c. Invers fungsi tangen

Fungsi f(x) = tanx, $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$ Monoton murni (selalu naik) sehingga mempunyai invers.




**Definisi** Invers dari tan x disebut fungsi arcus tanx, notasi arc tanx atau $\tan^{-1}(x)$

Berlaku hubungan

$$y = \tan^{-1} x \Leftrightarrow x = \tan y$$

Turunan

Dari $y = \tan^{-1} x \Leftrightarrow x = \tan y$ , $\frac{-\pi}{2} < y < \frac{\pi}{2}$

dan turunan fungsi invers diperoleh

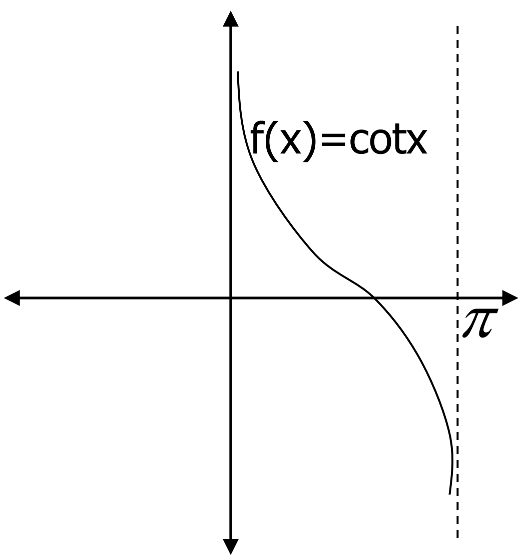
$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

atau $D_x(\tan^{-1} x) = \dfrac{1}{1+x^2}$ ⟹ $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

Jika u=u(x) $D_x(\tan^{-1} u) = \dfrac{u'}{1+u^2}$

d. Invers fungsi cotangen

Fungsi f(x)= cot x $,0 < x < \pi$ selalu monoton turun(monoton murni) sehingga mempunyai invers




**Definisi** Invers dari fungsi cot x disebut Arcus cotx, notasi arc cotx atau $\cot^{-1} x$

Berlaku hubungan

$$y = \cot^{-1} x \Leftrightarrow x = \cot y$$

Turunan

$$\frac{dy}{dx} = \frac{1}{dx/dy} = -\frac{1}{\csc^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

atau $D_x(\cot^{-1} x) = \frac{-1}{1+x^2}$ ⟹ $\int \frac{dx}{1+x^2} = -\cot^{-1} x + C$

Jika u=u(x) $D_x(\cot^{-1} u) = \frac{-u'}{1+u^2}$

Contoh

$$D_x(\tan^{-1}(x^2+1)) = \frac{1}{1+(x^2+1)^2} Dx(x^2+1) = \frac{2x}{1+(x^2+1)^2}$$

$$D_x(\cot^{-1}(\sin x)) = \frac{-1}{1+(\sin x)^2} Dx(\sin x) = \frac{-\cos x}{1+\sin^2 x}$$

Contoh Hitung

a. $\int \frac{dx}{4+x^2}$

b. $\int \frac{dx}{x^2+2x+4}$

Jawab

a. $\int \frac{1}{4+x^2}\,dx = \int \frac{1}{4(1+\frac{x^2}{4})}\,dx \quad = \frac{1}{4}\int \frac{1}{1+(\frac{x}{2})^2}\,dx$

$$u = \frac{x}{2} \rightarrow du = \tfrac{1}{2}\,dx \rightarrow dx = 2du$$

$$\int \frac{1}{4+x^2}\,dx = \frac{1}{4}\int \frac{2}{1+u^2}\,du = \frac{1}{2}\tan^{-1} u + C$$

$$= \frac{1}{2}\tan^{-1}(\frac{x}{2}) + C$$

Gunakan rumus

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C$$

b. $$\int \frac{dx}{x^2+2x+4} = \int \frac{1}{(x+1)^2+3}\,dx = \int \frac{1}{3(1+\frac{(x+1)^2}{3})}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1+\left(\frac{(x+1)}{\sqrt{3}}\right)^2}\,dx$$

Misal $$u = \frac{x+1}{\sqrt{3}} \rightarrow du = \frac{1}{\sqrt{3}}\,dx \rightarrow dx = \sqrt{3}\,du$$

Gunakan rumus

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}(u) + C$$

$$\int \frac{dx}{x^2+2x+4} = \frac{1}{3}\int \frac{\sqrt{3}}{1+u^2}\,du = \frac{1}{\sqrt{3}}\tan^{-1} u + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

# e. Invers fungsi secan

Diberikan f(x) = sec x $,0 \le x \le \pi, x \ne \frac{\pi}{2}$

$$f'(x) = \sec x \tan x > 0, 0 \le x \le \pi, x \ne \frac{\pi}{2}$$

⇓

f(x) = sec x monoton murni

⇓

Ada inversnya

**Definisi** Invers dari fungsi sec x disebut arcus secx, notasi arc secx atau $\sec^{-1} x$

Sehingga

$$y = \sec^{-1} x \Leftrightarrow x = \sec y$$

# Turunan

Dari $y = \sec^{-1} x \Leftrightarrow x = \sec y$ $\qquad$ $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$

$$\cos y = \frac{1}{x} \quad \Rightarrow \quad y = \cos^{-1}\left(\tfrac{1}{x}\right)$$

Sehingga

$$D_x(\sec^{-1} x) = D_x(\cos^{-1}\left(\tfrac{1}{x}\right)) = \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\,\frac{-1}{x^2} = \frac{|x|}{x^2\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

Jika u = u(x) $\quad D_x(\sec^{-1} u) = \dfrac{u'}{|u|\sqrt{u^2-1}}$

$$\Rightarrow \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + c$$

## e. Invers fungsi cosecan

Diberikan f(x) = csc x $, \frac{-\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0$

$$f'(x) = -\csc x \cot x < 0, \frac{-\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0$$

⇓

f(x) = sec x monoton murni

⇓

Ada inversnya

**Definisi** Invers dari fungsi csc x disebut arcus csc x, notasi arc cscx atau $\csc^{-1} x$

Sehingga

$$y = \csc^{-1} x \Leftrightarrow x = \csc y$$

## Turunan

Dari $y = \csc^{-1} x \Leftrightarrow x = \csc y$ $\qquad$ $\csc^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$

$$\sin y = \frac{1}{x} \quad \Rightarrow \quad y = \sin^{-1}\left(\frac{1}{x}\right)$$

Sehingga

$$D_x(\csc^{-1} x) = D_x(\sin^{-1}\left(\tfrac{1}{x}\right)) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \frac{-1}{x^2} = \frac{-|x|}{x^2\sqrt{x^2-1}} = \frac{-1}{|x|\sqrt{x^2-1}}$$

Jika u = u(x) $\qquad D_x(\sec^{-1} u) = \dfrac{-u'}{|u|\sqrt{u^2-1}}$

$$\Rightarrow \int \frac{1}{x\sqrt{x^2-1}}\,dx = -\csc^{-1}|x| + c$$

Contoh

A. Hitung turunan pertama dari

a. $f(x) = \sec^{-1}(x^2)$

b. $f(x) = \sec^{-1}(\tan x)$

Jawab

a. $$f'(x) = \frac{1}{|x^2|\sqrt{(x^2)^2 - 1}} Dx(x^2) = \frac{2x}{x^2\sqrt{x^4 - 1}} = \frac{2}{x\sqrt{x^4 - 1}}$$

b. $$f'(x) = \frac{1}{|\tan^2 x|\sqrt{(\tan x)^2 - 1}} Dx(\tan x) = \frac{\sec^2 x}{|\tan^2 x|\sqrt{\tan^2 x - 1}}$$

B. Hitung

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

Jawab

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{1}{x\sqrt{4(\frac{x^2}{4}-1)}}\,dx = \frac{1}{2}\int \frac{1}{x\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx$$

Misal $u = \dfrac{x}{2} \rightarrow du = \frac{1}{2}\,dx \rightarrow dx = 2du$

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx = \frac{1}{2}\int \frac{1}{2u\sqrt{u^2-1}}\,2du = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du$$

$$= \frac{1}{2}\sec^{-1}|u| + C = \frac{1}{2}\sec^{-1}\left|\frac{x}{2}\right| + C$$

## Soal Latihan

A. Tentukan turunan pertama fungsi berikut, sederhanakan jika mungkin

1. $y = (\sin^{-1} x)^2$

2. $y = \tan^{-1}(e^x)$

3. $y = \tan^{-1} x \ln x$

4. $f(t) = e^{\sec^{-1} t}$

5. $y = x^2 \cot^{-1}(3x)$

6. $y = \tan^{-1}(x - \sqrt{1 + x^2})$

B. Hitung

1. $\int \frac{dx}{9x^2+16}$

2. $\int \frac{dx}{4x\sqrt{x^2-16}}$

3. $\int \frac{dx}{\sqrt{2-5x^2}}$

4. $\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx$

5. $\int \frac{e^x}{e^{2x}+1} dx$

6. $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$

7. $\int \frac{dx}{x[4+(\ln x)^2]}$

# 8.8 Fungsi Hiperbolik

Definisi

a. Fungsi kosinus hiperbolik : $f(x) = \cosh x = \dfrac{e^{x} + e^{-x}}{2}$

b. Fungsi sinus hiperbolik : $f(x) = \sinh x = \dfrac{e^{x} - e^{-x}}{2}$

c. Fungsi tangen hiperbolik : $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

d. Fungsi cotangen hiperbolik : $f(x) = \coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

e. Fungsi secan hiperbolik : $f(x) = \sec hx = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x} - e^{-x}}$

f. Fungsi cosecan hiperbolik : $f(x) = \csc hx = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x} + e^{-x}}$

## Persamaan identitas pada fungsi hiperbolik

1. $\cosh x + \sinh x = e^{x}$
2. $\cosh x - \sinh x = e^{-x}$

3. $\cosh^2 x - \sinh^2 x = 1$

4. $1 - \tanh^2 x = \sec h^2 x$
5. $\coth^2 x - 1 = \csc h^2 x$

## Turunan

$$D_x(\cosh x) = D_x\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x \Rightarrow \int \sinh x\, dx = \cosh x + C$$

$$D_x(\sinh x) = D_x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x \Rightarrow \int \cosh x\, dx = \sinh x + C$$

$$D_x(\tanh x) = D_x\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \sec h^2 x$$

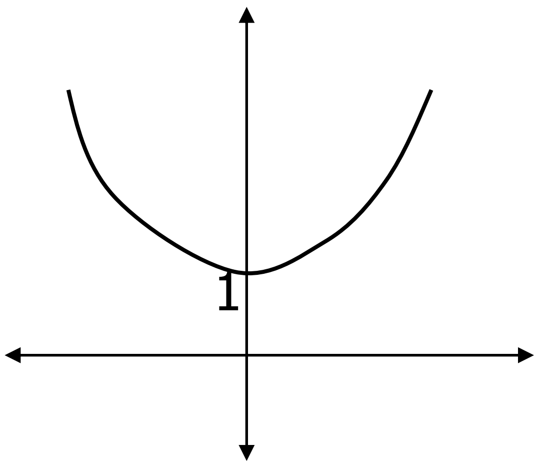
$$D_x(\coth x) = D_x\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \frac{-(\cosh^2 x - \sinh^2 x)}{\sinh^2 x}$$

$$= \frac{-1}{\sinh^2 x} = -\csc h^2 x$$

$$D_x(\sec hx) = D_x\left(\frac{1}{\cosh x}\right) = \frac{-\sinh x}{\cosh^2 x} = -\sec hx \tanh x$$

$$D_x(\csc hx) = D_x\left(\frac{1}{\sinh x}\right) = \frac{-\cosh x}{\sinh^2 x} = -\csc hx \coth x$$

# Grafik f(x) = coshx




Diketahui

(i) $f(x) = \cosh x = \dfrac{e^x + e^{-x}}{2}, x \in R$

(ii) $f'(x) = \dfrac{e^x - e^{-x}}{2} = \begin{cases} f'(x) < 0, & x < 0 \\ f'(x) > 0, & x > 0 \end{cases}$

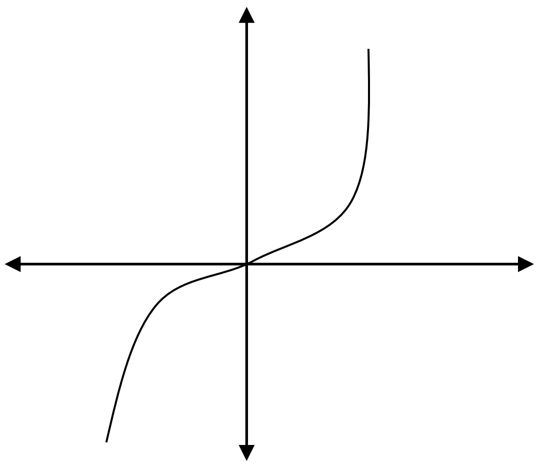
f monoton naik pada x > 0
monoton turun pada x < 0

(iii) $f''(x) = \dfrac{e^x + e^{-x}}{2} > 0, \forall x \in R$

Grafik f selalu cekung keatas

(iv) $f(0) = 1$

# Grafik f(x) = sinhx



Diketahui

(i) $f(x) = \sinh x = \dfrac{e^{x} - e^{-x}}{2}, x \in R$

(ii) $f'(x) = \dfrac{e^{x} + e^{-x}}{2} > 0$

f selalu monoton naik

(iii) $f''(x) = \dfrac{e^{x} - e^{-x}}{2} = \begin{cases} > 0, x > 0 \\ < 0, x < 0 \end{cases}$

Grafik f cekung keatas pada x>0
cekung kebawah pada x<0

(iv) $f(0) = 0$

Contoh

Tentukan $y'$ dari

1. $y = \tanh(x^2 + 1)$

2. $x^2 \sinh x + y^2 = 8$

Jawab

1. $y' = \text{sech}^2(x^2 + 1)\, Dx(x^2 + 1) = 2x\,\text{sech}^2(x^2 + 1)$

2. $Dx(x^2 \sinh x + y^2) = Dx(8)$

$$2x\sinh x + x^2 \cosh x + 2y\, y' = 0$$

$$y' = -\frac{2x\sinh x + x^2 \cosh x}{2y}$$

## Soal Latihan

A. Tentukan turunan pertama dari

1. $f(x) = \tanh 4x$
2. $g(x) = \sinh^2 x$
3. $g(x) = \dfrac{1 - \cosh x}{1 + \cosh x}$
4. $h(t) = \coth \sqrt{1 + t^2}$
5. $g(t) = \ln(\sinh t)$
6. $f(x) = x \cosh x^2$

B. Hitung integral berikut

1. $\int Sinh(1+4x)dx$

2. $\int \sinh x \cosh^2 x \ dx$

3. $\int \tanh x \ dx$

4. $\int \frac{\text{sech}^2 x}{2+\tanh x} dx$