Chapter 4
Time-Dependent Failure Models

- 4.1 The Weibull Distribution
- 4.2 The Normal Distribution
- 4.3 The Lognormal Distribution

4.1 The Weibull Distribution

$$\lambda(t) = at^{b}$$

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \qquad \theta > 0,\ \beta > 0,\ t \ge 0 \qquad (4.1)$$

$$R(t) = \exp\left[-\int_0^t \frac{\beta}{\theta}\left(\frac{t'}{\theta}\right)^{\beta-1} dt'\right] \qquad (4.2)$$

$$= e^{-(t/\theta)^{\beta}}$$

and $$f(t) = -\frac{dR(t)}{dt} = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} e^{-(t/\theta)^{\beta}} \qquad (4.3)$$

$$MTTF = \theta\Gamma\left(1 + \frac{1}{\beta}\right) \qquad (4.4)$$

and $$\sigma^2 = \theta^2\left\{\theta\Gamma\left(1 + \frac{2}{\beta}\right) - \left[\theta\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2\right\} \qquad (4.5)$$

where Γ(x) is the gamma function:

$$\Gamma(\mathrm{x}) = \int_0^\infty y^{x-1} e^{-y} dy$$

$$\text{for x} > 0,\ \Gamma(\mathrm{x}) = (x-1)\,\Gamma(\mathrm{x}-1)$$

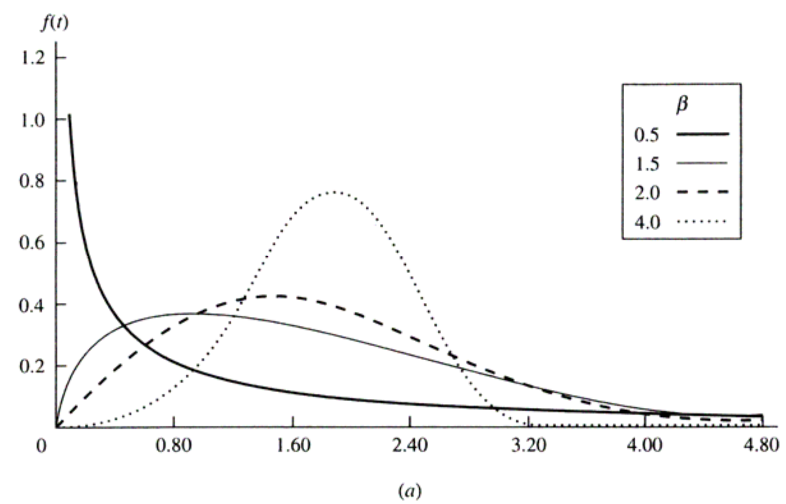

Figure 4.1a

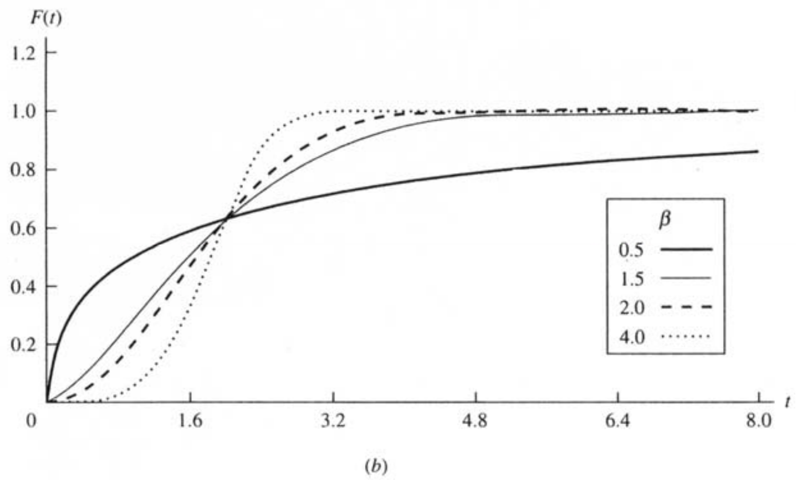

Figure 4.1b

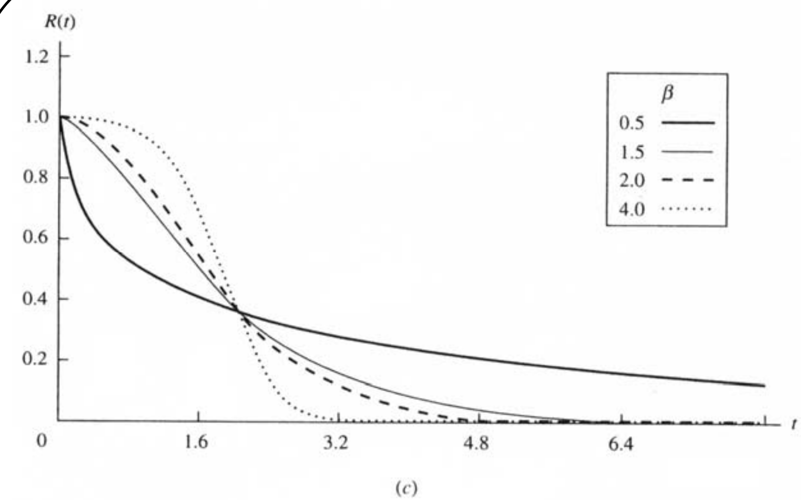

Figure 4.1c

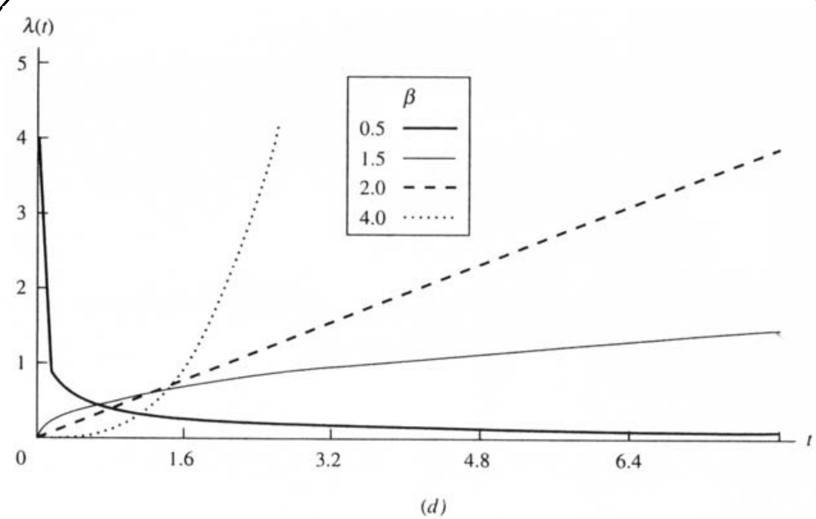

Figure 4.1d

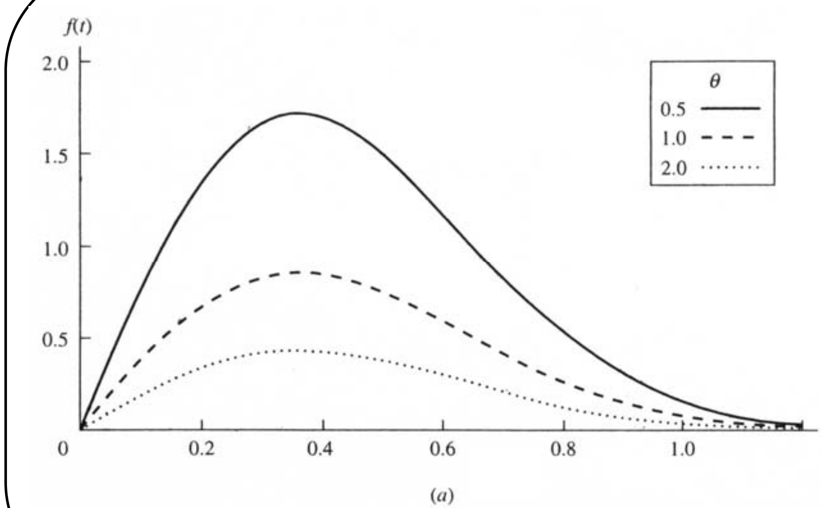

Figure 4.2a

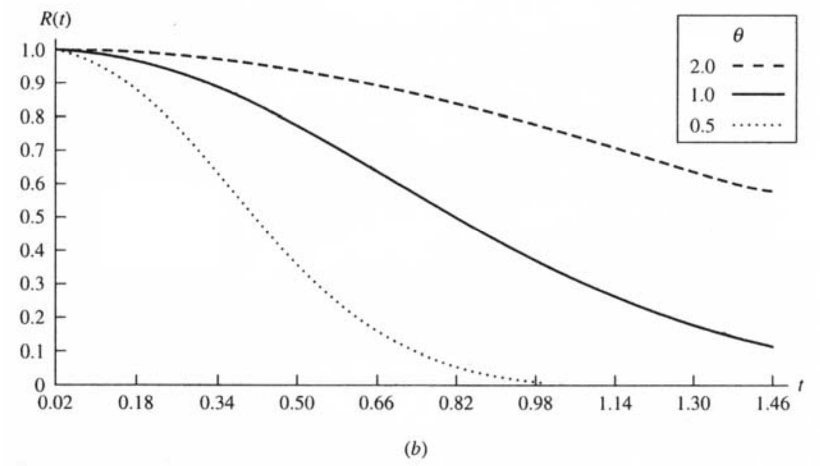

Figure 4.2b

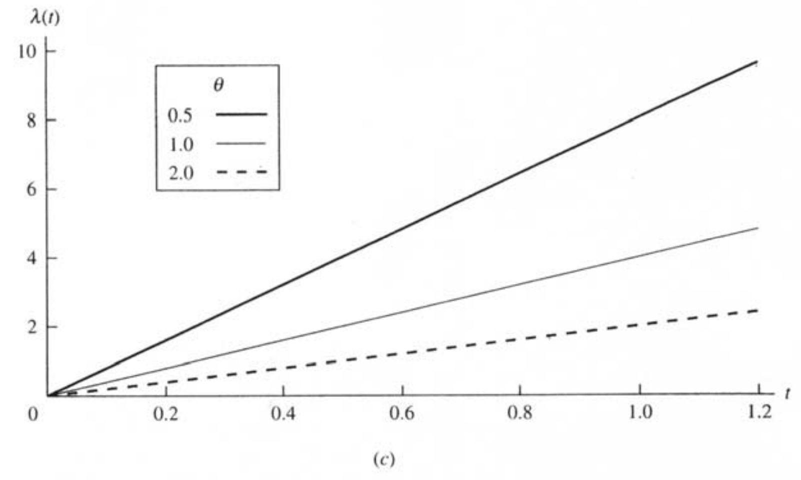

Figure 4.2c

Table 4.1 Weibull Parameter	
Value	**Property**
0<ß<1	Decreasing failure rate (DFR)
ß=1	Exponential distribution (CFR)
1<ß<2	IFR, concave
ß=2	Rayleigh distribution (LFR)
ß>2	IFR, convex
3≤ß≤4	IFR, Approaches normal distribution; symmetrical

Example 4.1 A compressor with a hazard rate function

$$\lambda(t) = \frac{2}{1000}\left(\frac{t}{1000}\right) = 2\times10^{-6}t$$

In this case, ß=2, θ=1000 hr.

For a desired 0.99 reliability:

$$R(t) = e^{-(t/1000)^2} = 0.99$$

The design life is given by

$$t_R = 1000\sqrt{-\ln 99} = 100.25 \text{ hr}$$

From Eqs. (4.4) and (4.5),

$$MTTF = 1000\,\Gamma\left(1+\frac{1}{2}\right) = 886.23 \text{ hr}$$

and $\sigma^2 = 10^6 \left\{ \Gamma(1+1) - \left[\Gamma\left(1+\frac{1}{2}\right) \right]^2 \right\}$

$= 214{,}601.7$

or $\sigma = 463.25$ hr

where $\Gamma\left(1+\frac{1}{2}\right) = 0.886227$

From Table A.9 in the Appendix.

$$\Gamma(2) = (2-1)! = 1$$

4.1.1 Design Life, Median and Mode

$$R(t) = e^{-(t/\theta)^\beta} = \text{R}$$

The design life is:

$$t_R = \theta(-\ln R)^{1/\beta} \quad (4.6)$$

The median time to failure is obtained when R = 0.5,

$$t_{0.50} = t_{\text{med}} = \theta(-\ln 0.5)^{1/\beta}$$

$$= \theta(0.69315)^{1/\beta} \quad (4.7)$$

Note: B1 life = $t_{0.99}$, B.1 life = $t_{0.999}$

The mode of the distribution: t *

$$f(\text{t}^*) = \max_{t \geq 0} f(t)$$

which results in

$$t_{\text{mode}} = \begin{cases} \theta[1 - (1/\beta)]^{1/\beta} & \text{for } \beta > 1 \\ 0 & \text{for } \beta \leq 1 \end{cases} \quad (4.8)$$

Example 4.2 A failure process as a Weibull failure distribution with a shape parameter = 1/3 and a scale parameter = 16,000.

Solution:

1. The reliability function is

$$R(t) = \exp\left[-\left(\frac{t}{16{,}000}\right)^{1/3}\right]$$

2. β = 1/3, a decreasing failure rate indicating high infant mortality.

3. MTFF $= 16{,}000\ \Gamma\left(1 + \frac{1}{1/3}\right) = 16{,}000 \cdot 3! = 96{,}000$ hr

Since the distribution is highly skewed, the median provides a better average. The mode is zero because ß<1.

4. $\sigma^2 = (16{,}000)^2 \{\Gamma(7) - [\Gamma(4)]^2\} = 175{,}104 \times 10^6$

$$\sigma = 418.4 \times 10^3 \text{ hr}$$

5. The characteristic life is 16,000 hr. Therefore 63 percent of the failures will occur by this time.

6. If a 90 percent reliability is desired, the design life is

$$t_G = 16{,}000\,(-\ln 0.90)^3 = 18.71 \text{ hr}$$

7. Its B1 life is (16,000)(-ln 0.99)^3 = 0.0162 hr, indicating a high percentage of early failures.

4.1.2 Burn-In Screening for Weibull

$$R(t|T_0) = \frac{\exp\{-[(t+T_0)/\theta]^\beta\}}{\exp\{-(T_0/\theta)^\beta\}} = \exp\left[-\left(\frac{t+T_0}{\theta}\right)^\beta + \left(\frac{T_0}{\theta}\right)^\beta\right]$$

For a 10-hr burn-in period:

$$R(t|10) = \exp\left[-\left(\frac{t+10}{16{,}000}\right)^{1/3} + \left(\frac{10}{16{,}000}\right)^{1/3}\right]$$

For R =0.90, the design life t_R can be obtained:

$$R(t_R|10) = 0.90$$

$$t_R = 16{,}000\left[-\ln 0.90 + \left(\frac{10}{16{,}000}\right)^{1/3}\right]^3 - 10 = 101.24 \text{ hr}$$

4.1.3 Failure Modes

For a system consisting of n components connected in series or with n independent failure modes. Each mode is an independent Weibull failure distribution with β and θ_i .

The system failure function is:

$$\lambda(t) = \sum_{i=1}^{n} \frac{\beta}{\theta_i}\left(\frac{t}{\theta_i}\right)^{\beta-1} = \beta t^{\beta-1}\left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^\beta\right] \qquad (4.9)$$

with shape parameter = β, and characteristic life is:

$$\theta = \left[\sum_{i=1}^{n}\left(\frac{1}{\theta_i}\right)^\beta\right]^{-1/\beta}$$

Example 4.3 A system consists of five modules, where each has a Weibull failure distribution:

$$\theta = \left[\left(\frac{1}{3600}\right)^{1.5} + \left(\frac{1}{7200}\right)^{1.5} + \left(\frac{1}{5850}\right)^{1.5} + \left(\frac{1}{4780}\right)^{1.5} + \left(\frac{1}{9300}\right)^{1.5}\right]^{-1/1.5} = 1842.7$$

and $MTTF = 1842.7\,\Gamma\left(1 + \frac{2}{3}\right) = 1664.5$ cycles

$$t_{med} = (1842.7)(0.69315)^{1/1.5} = 1443.2 \text{ cycles}$$

The reliability function for the engine is given by

$$R(t) = \exp\left[-\left(\frac{t}{1842.7}\right)^{1.5}\right]$$

4.1.4 Identical Weibull Components

For a system consisting of n serially connected and independent components with identical failure rate functions.

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} \qquad (4.10)$$

System failure rate is $\lambda(t) = \sum_{i=1}^{n} \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1} = \frac{n\beta}{\theta^{\beta}}(t)^{\beta-1}$

and $$R(t) = \exp\left[-n\left(\frac{t}{\theta}\right)^{\beta}\right] \qquad (4.11)$$

Example 4.4 $R(150) = \exp\left[-4\left(\frac{150}{2000}\right)^{3/4}\right] = 0.5637$

Example 4.4 A system with 4 series connectors each with a Weibull failure rate with $\beta = 3/4$ and $\theta = 2000$ hr.

System reliability at 150 hours of operation is:

$$R(150) = \exp\left[-4\left(\frac{150}{2000}\right)^{3/4}\right] = 0.5637$$

4.1.5 The Three-Parameter Weibull

t_0 = minimum life, where $T > t_0$

This three-parameter Weibull distribution assumes that no failures exist prior to time t_0.

$$R(t) = \exp\left[-\left(\frac{t-t_0}{\theta}\right)^{\beta}\right] \quad t \geq t_0 \qquad (4.12)$$

and $$\lambda(t) = \frac{\beta}{\theta}\left[-\left(\frac{t-t_0}{\theta}\right)^{\beta-1}\right] \quad t \geq t_0 \qquad (4.13)$$

t_0= location parameter.

The variance of this distribution is the same as for the 2-parameter model, but

$$MTTF = t_0 + \theta\Gamma\left(1 + \frac{1}{\beta}\right) \qquad (4.14)$$

$$t_{med} = t_0 + \theta(0.69315)^{1/\beta} \qquad (4.15)$$

and the design life t_R is

$$t_R = t_0 + \theta(-\ln R)^{1/\beta} \qquad (4.16)$$

To transform the 3-parameter Weibull into the 2-parameter Weibull: $t' = t - t_0$

Example 4.5: The 3-parameter Weibull has β=4, t_0 = 100, and θ = 780.

We obtain:

$$MTTF = 100 + 780\Gamma\left(1+\frac{1}{4}\right) = 806.99 \text{ hr}$$

$$t_{med} = 100 + 780(0.69315)^{1/4} = 811.7 \text{ hr}$$

$$\sigma^2 = (780)^2\left\{\Gamma\left(1+\frac{2}{4}\right) - \left[\Gamma\left(1+\frac{1}{4}\right)\right]^2\right\} = 39{,}384.6$$

$$\sigma = 198.3 \text{ hr}$$

$$R(500) = \exp\left[-\left(\frac{500-100}{780}\right)^4\right] = 0.933$$

4.1.6 Redundancy with Weibull Failures

For a system with 2 identical and independent components connected in parallel, the system reliability is:

$$R_s(t) = 1 - [1 - R(t)]^2$$

If $R(t) = e^{-(t/\theta)^\beta}$

then $R_s(t) = 1 - \left[1 - e^{-(t/\theta)^\beta}\right]^2 = 2e^{-(t/\theta)^\beta} - e^{-2(t/\theta)^\beta} \quad (4.17)$

and

$$MTTF = \int_0^\infty R_s(t)dt = \int_0^\infty \left[2e^{-(t/\theta)^\beta} - e^{-2(t/\theta)^\beta}\right]dt$$

$$= 2\int_0^\infty e^{-(t/\theta)^\beta}dt - \int_0^\infty e^{-2(t/\theta)^\beta}dt$$

The result is

$$MTTF = 2\theta\Gamma\left(1+\frac{1}{\beta}\right) - \frac{\theta}{2^{1/\beta}}\Gamma\left(1+\frac{1}{\beta}\right)$$

$$= \theta\Gamma\left(1+\frac{1}{\beta}\right)\left(2 - 2^{-1/\beta}\right) \quad (4.18)$$

The failure rate for this system (derived in App 4D) is:

$$\lambda_s(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}\frac{2-2e^{-(t/\theta)^\beta}}{2-e^{-(t/\theta)^\beta}} \quad (4.19)$$

The system does not have a Weibull failure rate. However, when *t* is large, the failure rate is approximated as a Weibull distribution.

Example 4.6 two fuel pumps, each has a Weibull failure distribution with β = ½ and θ = 1000 hr.

Find system reliability for 100-hr operation and the system MTTF.

Solution:

$$R_s(t=100) = 2\exp\left[-\left(\frac{100}{1000}\right)^{1/2}\right] - \exp\left[-2\left(\frac{100}{1000}\right)^{1/2}\right] = 0.9265$$

$$MTTF = 1000\Gamma(3)\left(2-2^{-2}\right) = 3500 \text{ hr}$$

Note: A fuel pump has $R_s(t) = 0.7288$, $MTTF = 2000$

4.2 The Normal Distribution

$$f(t)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(t-\mu)^2}{\sigma^2}\right] \qquad -\infty < t < \infty \quad (4.20)$$

$$R(t)=\int_t^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\frac{(t'-\mu)^2}{\sigma^2}\right]dt \quad (4.21)$$

$$z=\frac{T-\mu}{\sigma}$$

$$\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-z^2/2} \quad (4.22)$$

$$\Phi(z)=\int_{-\infty}^{z}\varphi(z')dz' \quad (4.23)$$

$$F(t)=\Pr\{T\le t\}=\Pr\left\{\frac{T-\mu}{\sigma}\le\frac{t-\mu}{\sigma}\right\}$$

$$=\Pr\left\{z\le\frac{t-\mu}{\sigma}\right\}=\Phi\left(\frac{t-\mu}{\sigma}\right) \quad (4.24)$$

$$R(t)=1-\Phi\left(\frac{t-\mu}{\sigma}\right) \quad (4.25)$$

with $F(t)=\Phi\left(\frac{t-\mu}{\sigma}\right)$

$$\lambda(t)=\frac{F(t)}{R(t)}=\frac{F(t)}{1-\Phi((t-\mu)/\sigma)} \quad (4.26)$$

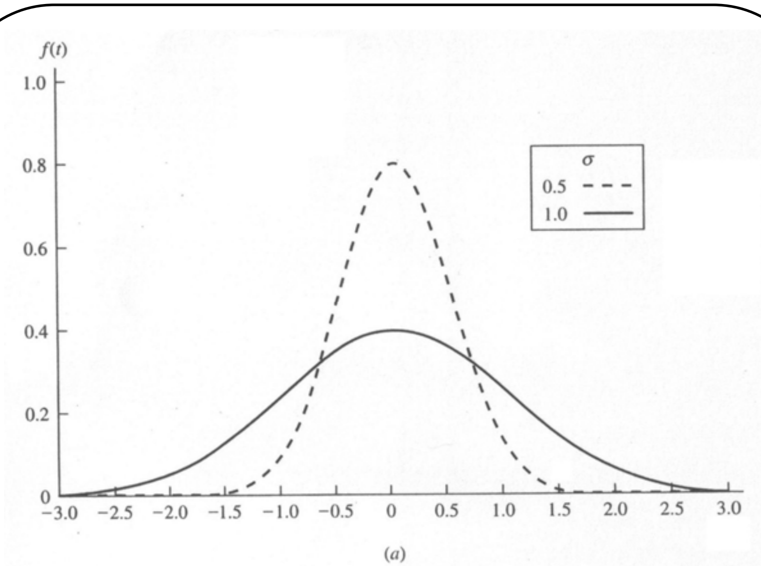

Figure 4.3a

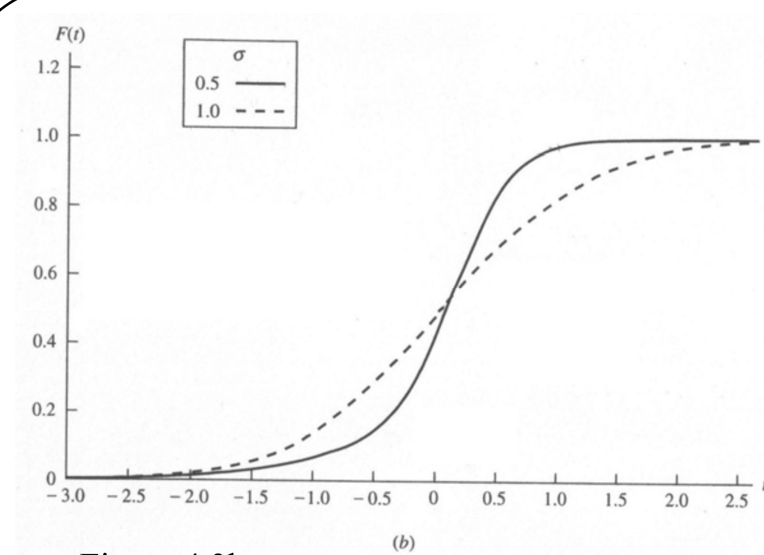

Figure 4.3b

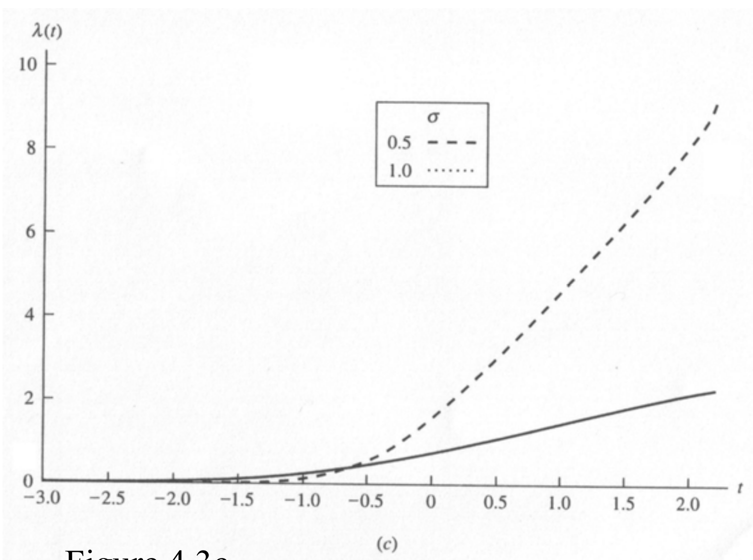

Figure 4.3c

4.3 The Lognormal Distribution

$$f(t)=\frac{1}{\sqrt{2\pi}st}\exp\left[-\frac{1}{2s^2}\left(\ln\frac{t}{t_{med}}\right)^2\right] \qquad t\geq 0 \qquad (4.27)$$

The mean, variance and the mode of the lognormal are:

$$MTTF = t_{med}^2\ \exp\left[s^2/2\right] \qquad (4.28)$$

$$\sigma^2 = t_{med}^2\ \exp\left(s^2\right)\left[\exp\left(s^2\right)-1\right] \qquad (4.29)$$

$$t_{\text{mod}e} = \frac{t_{med}}{\exp\left(s^2\right)} \qquad (4.30)$$

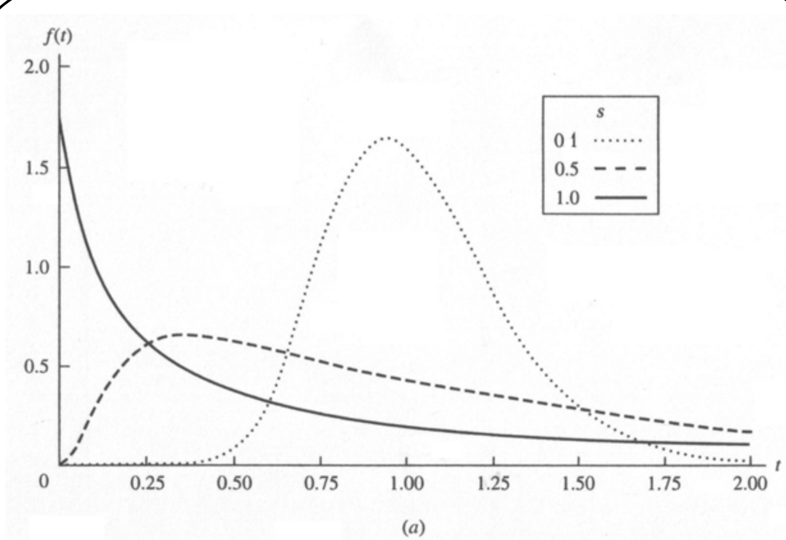

Figure 4.4a

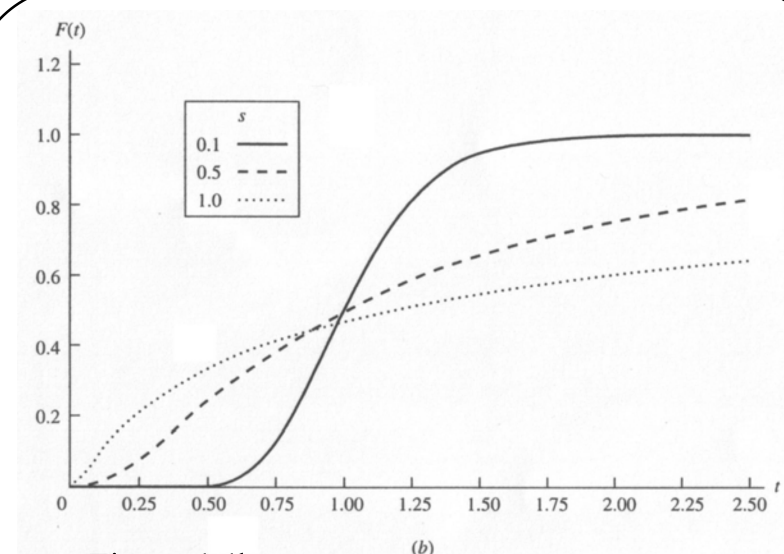

Figure 4.4b

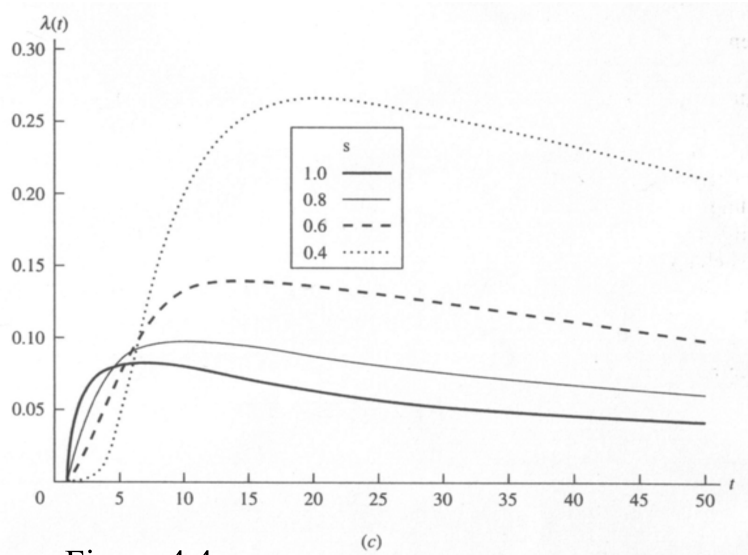

Figure 4.4c

Example 4.7

Solution:

Find $t_{0.95}$ such Pr $\{T \geq t_{0.95}\}$ = 0.95. Standardizing,

$$\Pr\left\{z \geq \frac{t_{0.95}-120}{14}\right\} = 1-\Phi\left(\frac{t_{0.95}-120}{14}\right)$$

Using the normal tables: $(t_{0.95}-120)/14 = -1.645$, or $t_{0.95}$=96.97 hr ≈ 8 (12-hr) days.

Example 4.8

Solution:

We are given Pr $\{25,000 \leq T \leq 35,00\}$ = 0.905. Standardizing,

$$\Pr\left\{\frac{25,000-\mu}{\sigma} \leq z \leq \frac{35,000-\mu}{\sigma}\right\} = 0.90$$

From the normal tables and the symmetry of the distribution,

$$\Pr\{-1.645 \leq z \leq 1.645\} = 0.90$$

or $\frac{25,000-\mu}{\sigma} = -1.645$ $\qquad$ $\frac{35,000-\mu}{\sigma} = 1.645$

Solving, $\mu = 30,000$ $\qquad$ $\sigma = 3039.5$

Therefore

$$R(24,000) = 1-\Phi\left(\frac{24,000-30,000}{3039.5}\right)$$
$$= 1-\Phi(-1.97) = 0.9756$$