START
Selected Topics in Assurance Related Technologies

Understanding Series and Parallel Systems Reliability

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Introduction

Reliability engineers often need to work with systems having elements connected in parallel and series, and to calculate their reliability. To this end, when a system consists of a combination of series and parallel segments, engineers often apply very convoluted block reliability formulas and use software calculation packages. As the underlying statistical theory behind the formulas is not always well understood, errors or misapplications may occur.

The objective of this START sheet is to help the reader better understand the statistical reasoning behind reliability block formulas for series and parallel systems and provide examples of the practical ways of using them. This knowledge will allow engineers to more correctly use the software packages and interpret the results.

We start this START sheet by providing some notation and definitions that we will use in discussing non-repairable systems integrated by series or parallel configurations:

1. All the "n" system component lives (X) are Exponentially distributed:

$$F(T) = P\{X \leq T\} = 1 - e^{-\lambda T}; \ f(T) = \frac{\delta}{dt} F(T) = \lambda e^{-\lambda T}$$

2. Therefore, every i^{th} component $1 \leq i \leq n$ Failure Rate (FR) is constant ($\lambda_i(t) = \lambda_i$).
3. All "n" system components are identical; hence, FR are equal ($\lambda_i = \lambda$; $1 \leq i \leq n$).
4. All "n" components (and their failure times) are statistically independent:

$$P\{X_1 \text{ and } X_2 \text{ and } ...X_n > T\}$$

$$= P\{X_1 > T\} P\{X_2 > T\}...P\{X_n > T\}$$

5. Denote system mission time "T". Hence, any i^{th} component ($1 \leq i \leq n$) reliability "$R_i(T)$":

$$R_i(T) = P(X_i > T) = e^{-\lambda T} \Rightarrow \lambda = -\frac{\ln(R_i(T))}{T}$$

Summarizing, in this START sheet we consider the case where life is exponentially distributed (i.e., component FR is time independent). First, examples will be given using identical components, and then examples will be considered using components with different FR. Independent components are those whose failure does not affect the performance of any other system component. Reliability is the probability of a component (or system) of surviving its mission time "T." This allows us to obtain both, component and system FR, from their reliability specification.

We will first discuss series systems, then parallel and redundant systems, and finally a combination of all these configurations, for non-repairable systems and the case of exponentially distributed lives. Examples of analyses and uses of reliability, FR, and survival functions, to illustrate the theory, are provided.

Reliability of Series Systems of "n" Identical and Independent Components

A series system is a configuration such that, if any one of the system components fails, the entire system fails. Conceptually, a series system is one that is as weak as its weakest link. A graphical description of a series system is shown in Figure 1.
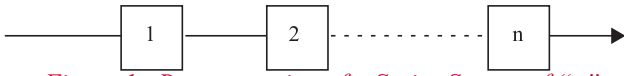
Figure 1. Representation of a Series System of "n" Components

Engineers are trained to work with system reliability [R_S] concepts using "blocks" for each system element, each block having its own reliability for a given mission time T:

$R_S = R_1 \times R_2 \times \ldots R_n$ (if the component reliabilities differ, or)

$R_S = [R_i]^n$ (if all i = 1, ... , n components are identical)

However, behind the reliability block symbols lies a whole body of statistical knowledge. For, in a series system of "n" components, the following are two equivalent "events":

"System Success" ≡ "Success of every individual component"

Therefore, the probability of the two equivalent events, that define total system reliability for mission time T (denoted R(T)), must be the same:

$$R(T) = P\{System \leftrightarrow Succeeds\} = P\{Comp1 \text{ and } Comp2...\text{and } Comp \text{ } n \leftrightarrow Succeed\}$$

$$= P\{Comp1 \leftrightarrow Suc\}...P\{Comp \text{ } n \leftrightarrow Suc\} = R_1(T)...R_n(T) = e^{-\lambda T}...e^{-\lambda T} = (e^{-\lambda T})^n$$

$$= [R_i(T)]^n = [Comp \text{ } Reliability(T)]^n = (e^{-\lambda})^{nT} = [R_i(1)]^{nT} = [Comp \text{ } Reliability(1)]^{nT}$$

The preceding assertion holds because $R_i(T)$, the probability of any component succeeding in mission time T, is its reliability. All system components are assumed identical with the same FR "λ" and independent. Hence, the product of all component reliabilities $R_i(T)$ yields the entire system reliability R(T). This allows us to calculate R(T) using system FR ($\lambda_s = n \times \lambda$), or the "n×T" power of unit time component reliability $[R_i(1)]^{nT}$, or the "n^{th}" power of component reliability $[R_i(T)]^n$, for any mission time T. We will discuss, later in this START sheet, the case where different components have different reliabilities or FR.

From all of the preceding considerations, we can summarize the following results when all elements, which are identical, of a system are connected in series:

1. The reliability of the entire system can be obtained in one of two ways:
   - $R(T) = [R_i(T)]^n$; i.e., the reliability (T) of any component "i" to the power "n"
   - $R(T) = [R_i(1)]^{nT}$; unit reliability of any component "i" to the power "nT"
2. System reliability can also be obtained by using system FR λ_s: $R(T) = \exp\{-\lambda_\sigma T\}$:
   - Since $\lambda_s = \lambda + \lambda + \lambda + \ldots + \lambda = n \times \lambda$ (all component FR λ are identical)

- System FR λ_s is then, the sum ("n" times) of all component failure rates (λ):

  $R(T) = Exp\{-(\lambda + \lambda + \lambda + \ldots + \lambda) \times T\} = Exp\{-n \times \lambda \times T\}) = Exp\{-\lambda_s T\}$

3. Component FR (λ) can be obtained from system reliability R(T):
   - $\lambda = [- \ln (R(T))] / n \times T$ (inverting the reliability results given in 1)
   - Component FR λ can also be obtained from component reliability $R_i(T)$:

     $\lambda = - \ln [R_i(T)]^n / n \times T = - \ln [R_i(T)] / T$
   - Previous expression is used for allocating system FR λ_s, among the system components
4. Total system FR λ_s can also be obtained from 3:
   - $\lambda_s = [- \ln (R(T))] / T = - \ln [R_i(T)]^n / T$
   - $\lambda_s = n \times \lambda$ remains time-independent in series configuration
5. Allocation of component reliability $R_i(T)$ from systems requirements is obtained by solving for $R_i(T)$ in the previous R(T) equations.
6. System "unreliability" = U(T) = 1 - R(T) = 1 - reliability.

One can calculate the various reliability and FR values for the special case of unit mission time (T = 1) by letting "T" vanish from all the formulas (e.g., substituting T by 1). One can obtain reliability R(T) for any mission time T, from R(1), reliability for unit mission time:

$$R(T) = P(X_1 ..., X_n > T) = e^{-\lambda_s T} = (e^{-\lambda_s})^T = [R(1)]^T$$

Numerical Examples

The concepts discussed are best explained and understood by working out simple numerical examples. Let a computer system be composed of five identical terminals in series. Let the required system reliability, for unit mission time (T = 1) be R(1) = 0.999.

We will now calculate each component's reliability, unreliability, and failure rate values.

From the data and formulas just given, each terminal reliability $R_i(T)$ can be obtained by inverting the system reliability R(T) equation for unit mission time (T = 1):

$$R(1) = e^{-\lambda_s} = (e^{-5\lambda}) = (e^{-\lambda})^5 = [R_i(1)]^5 = 0.999$$

$$\Rightarrow R_i(1) = [R(1)]^{1/5} = (0.999)^{1/5} = 0.9998$$

Component unreliability is: $U_i(1) = 1 - R_i(1) = 1 - 0.9998 = 0.0002$.

Component FR is obtained by solving for λ in the equation for component reliability:

$$\lambda = -\frac{\ln\left(R_i(T)\right)}{T} = \frac{-\ln(0.9998)}{1} = 0.0002$$

Now, assume, that component reliability for mission time $T = 1$ is given: $R_i(1) = 0.999$. Now, we are asked to obtain total system reliability, unreliability, and FR, for the (computer) system and mission time $T = 10$ hours. First, for unit time:

$$R(1) = e^{-\lambda_s} = (e^{-5\lambda}) = (e^{-\lambda})^5 = \left[R_i(1)\right]^5 = (0.999)^5 = 0.995$$

Hence, system FR is:

$$\lambda_s = -\frac{\ln\left(R(T)\right)}{T} = \frac{-\ln(0.995)}{1} = 0.005013$$

If we require system reliability for mission time $T = 10$ hours, $R(10)$, and the unit time reliability is $R(1) = 0.995$, we can use either the 10th power or the FR λ_s:

$$R(10) = e^{-10\lambda_s} = (e^{-\lambda_s})^{10} = \left[R(1)\right]^{10} = (0.995)^{10}$$

$$= e^{-10\lambda_s} = (e^{-10 \times 0.00501}) = e^{-0.05} = 0.9512$$

If mission time T is arbitrary, then $R(T)$ is called "Survival Function" (of T). $R(T)$ can then be used to find mission time "T" that accomplishes a pre-specified reliability. Assume that $R(T) = 0.98$ is required and we need to find out maximum time T:

$$R(T) = e^{-\lambda_s T} = e^{-n\lambda T} = 0.98; \Rightarrow T = -\frac{\ln R(T)}{\lambda_s} = -\frac{\ln(0.98)}{0.005013} = 4.03$$

Hence, a Mission Time of $T = 4.03$ hours (or less) meets the requirement of reliability 0.98 (or more).

Let's now assume that a new system, a ship, will be propelled by five identical engines. The system must meet a reliability requirement $R(T) = 0.9048$ for a mission time $T = 10$. We need to allocate reliability by engine (component reliability), for the required mission time T. We invert the formula for system reliability $R(10)$, expressed as a function of component reliability. Then, we solve for component reliability $R_i(10)$:

$$R(10) = e^{-10\lambda_s} = (e^{-10 \times 5\lambda}) = (e^{-10\lambda})^5 = \left[R_i(10)\right]^5 = 0.9048$$

$$\Rightarrow R_i(10) = \left[R(10)\right]^{1/5} = (0.9048)^{0.2} = 0.9802$$

We now calculate system FR (λ_s) and MTTF (μ) for the five-engine system. These are obtained for mission time $T = 10$ hours and required system reliability $R(10) = 0.9048$:

$$\lambda_s = -\frac{\ln\left(R(T)\right)}{T} = \frac{-\ln(0.9048)}{10} = \frac{0.1001}{10}$$

$$= 0.010005 \Rightarrow MTTF = \mu = \frac{1}{\lambda_s} = 99.96$$

FR and MTTF values, equivalently, can be obtained using FR per component, yielding the same results:

$$\lambda = -\frac{\ln\left(R_i(T)\right)}{T} = \frac{-\ln(0.9802)}{10} = \frac{0.019999}{10} = 0.0019999$$

$$\Rightarrow \lambda_s = \sum \lambda_i = 5 \times \lambda = 5 \times 0.0019999 = 0.009999 \approx 0.01$$

$$\Rightarrow MTTF = \int_0^\infty R(T)\,dT = \int_0^\infty e^{-\lambda_s T}\,dT = \mu = \frac{1}{\lambda_s} = 99.96$$

Finally, assume that the required ship FR $\lambda_s = 5 \times \lambda = 0.010005$ is given. We now need component reliability, Unreliability and FR, by unit mission time ($T = 1$):

$R(1) = \text{Exp}\{-\lambda_s\} = \text{Exp}\{-0.010005\} = 0.99$
$= \text{Exp}\{-5 \times \lambda\} = [\text{Exp}(-\lambda)]^5 = [R_i(1)]^5$

Component reliability: $R_i(1) = [R(1)]^{1/5} = [0.99]^{0.2}$
$= 0.998$

Component unreliability: $U_i(1) = 1 - R_i(1) = 1 - 0.998$
$= 0.002$

Component FR: $\lambda = [-\ln(R(1))]/n \times 1 = [-\ln(0.99)]/5$
$= 0.002$

The Case of Different Component Reliabilities

Now, assume that different system components have different reliabilities and FR. Then:

$$R(T) = R_1(T)\ldots R_n(T) = e^{-\lambda_1 T}\ldots e^{-\lambda_n T} = e^{-T\Sigma_i \lambda_i} = e^{-\lambda_s T} \Rightarrow \lambda_s = \Sigma_{i=1}^n \lambda_i$$

Then system Mean Time To Failure, MTTF, $= \mu = 1/\lambda_s = 1/\Sigma \lambda_i$

For example, assume that the five engines (components), in the above system (ship) have different reliabilities (maybe they come from different manufacturers, or exhibit different ages). Let their reliabilities, for mission time ($T = 10$) be 0.99, 0.97,

0.95, 0.93, and 0.9, respectively. Then, total system reliability $R(T)$ for $T = 10$ and FR are:

$$R(T) = R_1(T)..R_n(T) = 0.99 \times 0.97 \times 0.95 \times 0.93 \times 0.9 = 0.7636$$

$$\Rightarrow \lambda_s = -\frac{\ln(R(T))}{T} = \frac{-\ln\{R(10)\}}{10} = \frac{0.2697}{10} \, 0.02697$$

Since the system FR is $\lambda_s = 0.02697$, then the system MTTF is $\mu = 1/\lambda_\sigma = 1/\Sigma\,\lambda_i = 1/0.02697 = 37.077$.

Reliability of Parallel Systems

A parallel system is a configuration such that, as long as not all of the system components fail, the entire system works. Conceptually, in a parallel configuration the total system reliability is higher than the reliability of any single system component. A graphical description of a parallel system of "n" components is shown in Figure 2.
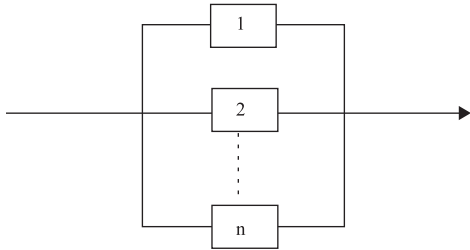

Figure 2. Representation of a Parallel System of "n" Components

Reliability engineers are trained to work with parallel systems using block concepts:

$R_S = 1 - \Pi\,(1 - R_i) = 1-(1 - R_1) \times (1 - R_2) \times \dots (1 - R_n)$;   if the component reliabilities differ, or

$R_S = 1 - \Pi\,(1 - R_i) = 1-[1 - R]^n$;   if all "n" components are identical: $[R_i = R;\ i = 1, \dots, n]$

However, behind the reliability block symbols lies a whole body of statistical knowledge. To illustrate, we analyze a simple parallel system composed of n = 2 identical components. The system can survive mission time T only if the first component, or the second component, or both components, survive mission time T (Figure 3). In the language of statistical "events":

$$R(T) = P\{\text{System Survives T}\} = P\{X_1 > T \text{ or } X_2 > T \text{ or BOTH} > T\}$$

$$= P\{X_1 > T\} + P\{X_2 > T\} - P\{X_1 > T \text{ and } X_2 > T\}$$

$$= R_1(T) + R_2(T) - R_1(T) \times R_2(T)$$

$$= R_1(T)[1 - R_2(T)] + R_2(T) + (1-1) = 1 + R_1(T)[1 - R_2(T)] - [1 - R_2(T)]$$

$$= 1 - [1 - R_1(T)][1 - R_2(T)] = 1 - [1 - P\{X_1 > T\}][1 - P\{X_2 > T\}]$$

$$= 1 - P\{X_1 \leq T\}\,P\{X_2 \leq T\}$$

$$= 1 - \left[1 - e^{-\lambda_1 T}\right]\left[1 - e^{-\lambda_2 T}\right];\ \text{If } R_1(T) \neq R_2(T) \Rightarrow \lambda_1 \neq \lambda_2$$

$$= 1 - [1 - P\{X > T\}]^2 = 1 - \left[1 - e^{-\lambda T}\right]^2;\ \text{If } R_1(T) = R_2(T)$$
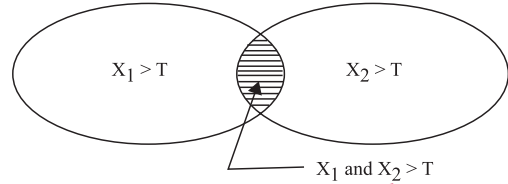

Figure 3. Venn Diagram Representing the "Event" of Either Device or Both Surviving Mission Time

This approach easily can be extended to an arbitrary number of "n" parallel components, identical or different. By expanding the formula $R_S = 1 -(1 - R_1) \times (1 - R_2) \times \dots (1 - R_n)$ into products, the well-known reliability block formulas are obtained. For example, for n = 3 blocks, when only one is needed:

$$R_S = 1 -(1 - R_1) \times (1 - R_2) \times (1 - R_3) = R_1 + R_2 + R_3 - R_1R_2 - R_1R_3 - R_2R_3 + R_1R_2R_3 \text{ or}$$

$$R_S = 1 -(1 - R) \times (1 - R) \times (1 - R) = 3R - 3R^2 + R^3 \text{ (if all components are identical: } R_i = R;\ i = 1, \dots, n$$

Using instead, the statistical formulation of the Survival Function $R(T)$, we can obtain system MTTF (μ) for an arbitrary mission time T. For, say n = 2 arbitrary components:

$$R(T) = 1 - [1 - R_1(T)][1 - R_2(T)] = 1 - \left[1 - e^{-\lambda_1 T}\right]\left[1 - e^{-\lambda_2 T}\right]$$

$$= e^{-\lambda_1 T} + e^{-\lambda_2 T} - e^{-(\lambda_1 + \lambda_2)T}$$

$$\Rightarrow \text{MTTF} = \mu = \int_0^\infty R(T)\,dT = \int_0^\infty \left(e^{-\lambda_1 T} + e^{-\lambda_2 T} - e^{-(\lambda_1 + \lambda_2)T}\right)dT$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$$

Finally, one can calculate system FR λ_s from the theoretical definition of FR. For n = 2:

$$\text{FR} = \lambda_s = \frac{\text{Density Function}}{\text{Survival Function}} = \frac{-\frac{\delta}{dt}R(T)}{R(T)}$$

$$= \frac{\lambda_1 e^{-\lambda_1 T} + \lambda_2 e^{-\lambda_2 T} - (\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)T}}{e^{-\lambda_1 T} + e^{-\lambda_2 T} - e^{-(\lambda_1 + \lambda_2)T}} \equiv \lambda_s(T)$$

Notice from this derivation that, even when every component FR(λ) is constant, the resulting parallel system Hazard Rate $\lambda_s(T)$ is time-dependent. This result is very important!

Numerical Examples

Let a parallel system be composed of n = 2 identical components, each with FR $\lambda = 0.01$ and mission time T = 10 hours, only one of which is needed for system success. Then, total system reliability, by both calculations, is:

$$R_i(10) = P\{X > 10\} = e^{-10\lambda} = e^{-0.1} = 0.9048; i = 1,2$$

$$R(10) = 1 - [1 - R_1(10)][1 - R_2(T)] = 1 - [1 - R_i(10)]^2$$

$$= 1 - (1 - 0.9048)^2 = 0.9909$$

$$R(T) = e^{-\lambda_1 T} + e^{-\lambda_2 T} - e^{-(\lambda_1 + \lambda_2)T} = 2e^{-\lambda T} - e^{-2\lambda T}$$

$$R(10) = 2e^{-10\lambda} - e^{-20\lambda} = 2e^{0.1} - e^{-0.2} = 0.9909; \text{ for } T = 10;$$

Mean Time to Failure (in hours):

$$MTTF = \mu = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2} = \frac{2}{0.01} - \frac{1}{0.02} = 150$$

The failure (hazard) rate for the two-component parallel system is now a function of T:

$$\lambda_s(T) = \frac{\lambda_1 e^{-\lambda_1 T} + \lambda_2 e^{-\lambda_2 T} - (\lambda_1 + \lambda_2)e^{-(\lambda_1 + \lambda_2)T}}{e^{-\lambda_1 T} + e^{-\lambda_2 T} - e^{-(\lambda_1 + \lambda_2)T}}$$

$$\equiv \frac{0.02e^{-0.01T} - 0.02e^{-0.02T}}{2e^{-0.01T} - e^{-0.02T}}$$

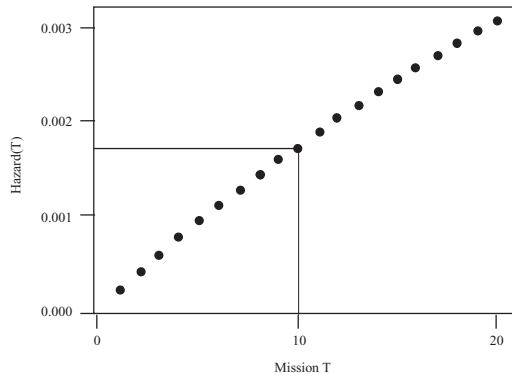This system hazard rate $\lambda_s(T)$ can be calculated as a function of any mission time T, as shown in Figure 4.

Figure 4. Plot of the Hazard $\lambda_s(T)$ as a Function of Mission Time T. Hazard Rate $\lambda_s(T)$ increases as time T increases. This plot can be used to find the $\lambda_s(T)$ required to meet a Mission Time of T. Say T = 10, then $\lambda_s(T)$ about 0.0018

Reliability of "K out of N" Redundant Systems with "n" Identical Components

A "k" out of "n" redundant system is a parallel configuration where "k" of the system components, as a minimum, are required to be fully operational at the completion time T of the mission, for the system to "succeed" (for k = 1 it reduces to a parallel system; for k = n, to a series one). We illustrate this using the example of a system operation depicted in Figure 5.

The Probability "p" for any system unit or component "i", $1 \leq i \leq n$, to survive mission time T is:

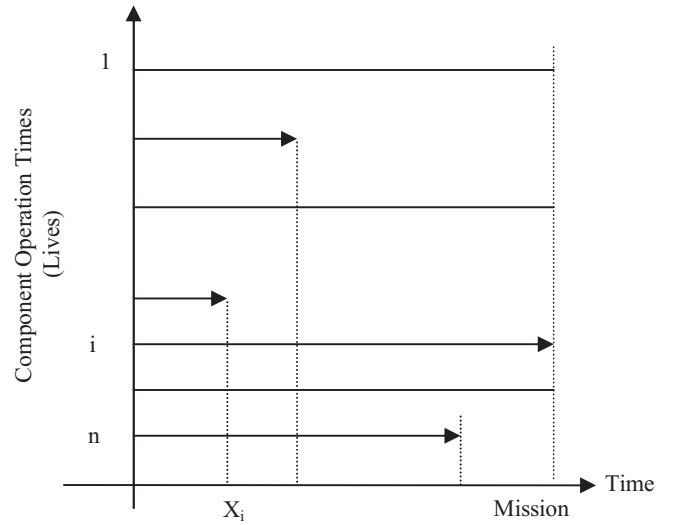$$R_i(T) = P(X_i > T) = e^{-\lambda T} = P$$

Figure 5. Units Either Fail/Survive Mission Time

All units are identical and "k" or more units, out of the "n" total, are required to be operational at mission time T, for the entire system to fulfill the mission. Therefore, the Probability of Mission Success (i.e., system reliability) is equivalent to the probability of obtaining "k" or more successes out of the possible "n" trials, with success probability p.

This probability is described by the Binomial (n, p) Distribution. In our case, the probability of success "p" is just the reliability $R_i(T)$ of any independent unit or component "i", for the required mission time "T". Therefore, total system reliability R(T), for an arbitrary mission time T, is calculated by:

$$R(T) = \Sigma_{j=k}^{n} P(\text{Succ.} = j; \text{Tot.} = n; \text{Unit Rel.} = p)$$

$$= \Sigma_{j=k}^{n} C_j^n p^j (1-p)^{n-j} = \Sigma_{j=k}^{n} B(j; n, p)$$

Sometimes the formula: $1 - \Sigma_{j=0}^{k-1} C_j^n p^j (1-p)^{n-j}$ is used instead. This holds true because:

$$1 = \Sigma_{j=0}^{k-1} C_j^n p^j (1-p)^{n-j} + \Sigma_{j=k}^{n} C_j^n p^j (1-p)^{n-j}$$

The "summation" values are obtained using the Binomial Distribution tables or the corresponding Excel algorithm (formula).

Following the same approach of the series system case, we obtain the MTTF (μ).

$$R(T) = \sum_{j=k}^{n} C_j^n e^{-\lambda Tj} (1 - e^{-\lambda T})^{n-j} \Rightarrow MTTF = \mu$$

$$= \int_0^\infty R(T)dt = \sum_{j=k}^{n} C_j^n \int_0^\infty e^{-\lambda Tj} (1 - e^{-\lambda T})^{n-j} \, dt = \frac{1}{\lambda} \sum_{j=k}^{n} \frac{1}{j}$$

We can obtain all parameters for an arbitrary T, by recalculating probability $p = e^{-\lambda T}$ of a component surviving this new mission time "T". In the special case of mission time T = 1, the "T" vanishes from all these formulas (e.g., substituted T by 1).

Applying the immediately preceding assumptions and formulas, we obtain the following results:

- The reliability R(T) of the entire system, for specified T, is obtained by:
  - Providing the total number of system components (n) and required ones (k)
  - Providing the reliability (for mission time T) of one component: $R_i(T) = p$
  - Alternatively, providing the Failure Rate (FR) λ of one unit or component
- System MTTF can be obtained from R(T) using the preceding inputs and:
  - $MTTF = \frac{1}{\lambda} \sum_{j=k}^{n} \frac{1}{j}$
- The "Unreliability" = U(T) = 1 - Reliability = 1 - R(T)

Numerical Example

Let there be n = 5 identical components (computers) in a system (shuttle). Define system "success" if k = 2 or more components (computers) are running during re-entry. Let every component (computer) have a reliability $R_i(1) = 0.9$. Let mission "re-entry" time be T = 1. If each component has a reliability $R_i(T) = p = 0.9$, then total system (shuttle) reliability R(T), the component FR (λ) and the MTTF (μ) are obtained as:

$$R(1) = \sum_{j=2}^{5} P(Succ. = j; Tot. = 5; Unit\ Rel. = 0.9)$$

$$= \sum_{j=2}^{5} C_j^n 0.9^j (1 - 0.9)^{5-j}$$

$$= 1 - \sum_{j=0}^{2-1} C_j^n 0.9^j (1 - 0.9)^{5-j}$$

$$= 1 - 0.00046 = 0.99954 = e^{-\lambda_s}$$

$$R_i(1) = 0.9 = e^{-\lambda} \Rightarrow \lambda = -\ln\{R_i(1)\}$$

$$= -\ln(0.9) = 0.105361$$

$$MTTF = \mu = \frac{1}{\lambda} \sum_{j=k}^{n} \frac{1}{j} = \frac{1}{0.105361} x \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right)$$

$$= 9.491 \ x \ 1.283 = 12.177$$

Now, assume that a less expensive design is being considered, consisting of n = 8 identical components in parallel. The new design requires that at least k = 5 units are working for a successful completion of the mission. Assume that mission time is T = 1 and the new component FR $\lambda = 0.223144$. Compare the two system reliabilities and MTTFs.

First, we need to obtain the new component reliability $R_i(T) = p$ for T = 1:

$$R_i(1) = P(X > 1) = e^{-\lambda} = e^{-0.223144} = 0.79999 \approx 0.8 = p$$

Proceeding as before, we obtain the new total system reliability for unit mission time:

$$R(1) = \sum_{k=5}^{8} C_j^n 0.8^j (1 - 0.8)^{8-j} = 1 - \sum_{j=0}^{5-1} C_j^n 0.8^j (1 - 0.8)^{8-j}$$

$$= 1 - 0.05628 = 0.94372 = e^{-\lambda_s}$$

$$MTTF = \mu = \frac{1}{\lambda} \sum_{j=k}^{n} \frac{1}{j} = \frac{1}{0.223144} x \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right)$$

$$= 4.481 \ x \ 1.283 = 5.7497$$

The cheaper (second) design is, therefore, less reliable (and has a lower MTTF) than the first design.

Combinations of Configurations

Some systems are made up of combinations of several series and parallel configurations. The way to obtain system reliability in such cases is to break the total system configuration down into homogeneous subsystems. Then, consider each of these subsystems separately as a unit, and calculate their reliabilities. Finally, put these simple units back (via series or parallel recombination) into a single system and obtain its reliability.

For example, assume that we have a system composed of the combination, in series, of the examples developed in the previous two sections. The first subsystem, therefore, consists of two identical components in parallel. The second subsystem consists of a "2 out of 5" (parallel) redundant configuration, composed of also five identical components (Figure 6). Assume also that Mission Time is T = 10 hours.
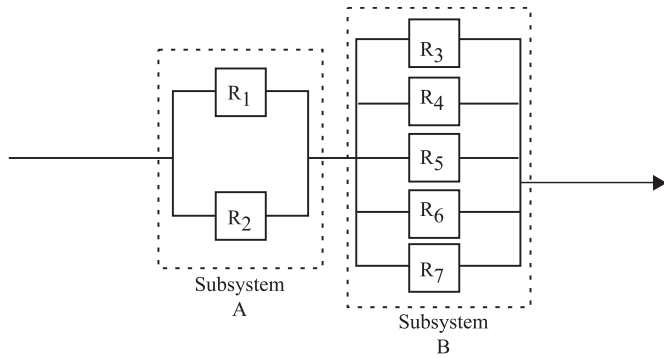
Figure 6. A Combined Configuration of Two Parallel Subsystems in Series

Using the same values as before, for subsystem, A (two identical components in parallel, with FR $\lambda = 0.01$ and mission time T = 10 hours), we can calculate reliability as:

$$R_A(10) = 1 - \left[1 - R_1(10)\right]\left[1 - R_2(10)\right]$$

$$= 1 - \left[1 - R_i(10)\right]^2 = 1 - (1 - 0.9048)^2 = 0.9909$$

Similarly, subsystem B ("2 out of 5" redundant) has five identical components, of which at least two are required for the subsystem mission success. $R_3(1) = R_4(1) = R_5(1) = R_6(1) = R_7(1) = 0.9$, for T = 1. We first recalculate the component reliability for the new mission time T = 10 and then calculate subsystem B reliability as follows:

$$R_i(1) = P\{X > 1\} = e^{-\lambda} = 0.9$$

$$\Rightarrow \lambda = -\ln\{R_i(1)\} = -\ln(0.9) = 0.105361$$

$$R_i(10) = P\{X > 10\} = e^{-\lambda T} = p$$

$$= e^{-0.105361 \times 10} = e^{-1.05361} = 0.3487 = p$$

$$R_B(10) = \sum_{j=2}^{5} P(\text{Succ.} = j; \text{Tot.} = 5; p = 0.3487)$$

$$= 1 - \sum_{j=0}^{2-1} C_j^n 0.3487^j (1 - 0.3487)^{5-j}$$

$$= 1 - 0.4309 = 0.5691 = e^{-10\lambda_s}$$

Recombining both subsystems, we get a series system, consisting of subsystems A and B. Therefore, the combined system reliability, for mission time T = 10, is:

$$R(10) = R_A(10) \times R_B(10) = 0.9909 \times 0.5691 = 0.5639$$

This result immediately shows which subsystem is driving down the total system reliability and sheds light about possible measures that can be taken to correct this situation.

Summary

The reliability analysis for the case of non-repairable systems, for configurations in series, in parallel, "k out of n" redundant and their combinations, has been reviewed for the case of exponentially-distributed lives. When component lives follow other distributions, we substitute the density function in the corresponding reliability formulas R(T) and redevelop the algebra. Of particular interest is the case when component lives have an underlying Weibull distribution:

$$F(T) = P\{X \le T\} = 1 - e^{-\left(\frac{T}{\alpha}\right)^{\beta}}$$

$$f(T) = \frac{\delta}{dt} F(T) = \frac{\beta}{\alpha^{\beta}} T^{\beta-1} e^{-\left(\frac{T}{\alpha}\right)^{\beta}}$$

Here, we substitute these values into equations 1 through 5 of the first section and 1 through 6 of the second section and redevelop the algebra. Due to its complexity, this case will be the topic of a separate START sheet. Finally, for those readers interested in pursuing these studies at a more advanced level, we provide a useful bibliography For Further Study.

For Further Study

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About the Author

Dr. Jorge Luis Romeu has over thirty years of statistical and operations research experience in consulting, research, and teaching. He was a consultant for the petrochemical, construction, and agricultural industries. Dr. Romeu has also worked in statistical and simulation modeling and in data analysis of software and hardware reliability, software engineering, and ecological problems.

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