Chapter 6
State-Dependent Systems

6.1 MARKOV ANALYSIS

When failure dependency exists in the system

= component failures are dependent

→ Markov model is a typical method to approach the problem

State	Component 1	Component 2
1	Work	Work
2	Fail	Work
3	Work	Fail
4	Fail	Fail

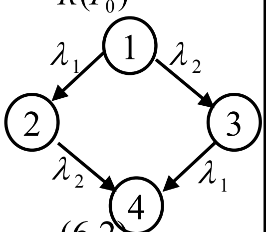
Memoryless Property

$$R(t \mid T_0) = P(T > T_0 + t \mid T > T_0) = \frac{P(T > T_0 + t)}{P(T > T_0)} = \frac{R(T_0 + t)}{R(T_0)} = R(t)$$

$P_1(t) + P_2(t) + P_3(t) + P_4(t) = 1$ (6.1)

$R_{series}(t) = P_1(t)$

$R_{parallel}(t) = P_1(t) + P_2(t) + P_3(t)$



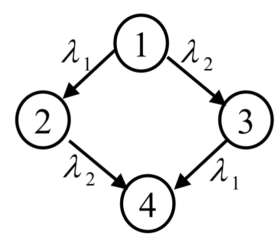
$P_1(t+\Delta t) = P_1(t) - \lambda_1 \Delta t P_1(t) - \lambda_2 \Delta t P_1(t)$ (6.2)

It is the probability of the system staying at state 1 at time t+ Δt which is equal to the prob of it being at state 1 at time t minus the prob of it being at state 1 times the prob of transitioning ($\lambda_i \Delta t$) to state 2 or 3.

$P_2(t+\Delta t) = P_2(t) + \lambda_1 \Delta t P_1(t) - \lambda_2 \Delta t P_2(t)$ (6.3)

$P_3(t+\Delta t) = P_3(t) + \lambda_2 \Delta t P_1(t) - \lambda_1 \Delta t P_3(t)$ (6.4)

$P_4(t+\Delta t) = P_4(t) + \lambda_2 \Delta t P_2(t) + \lambda_1 \Delta t P_3(t)$ (6.5)



From Eq.(6.2)

$$\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} = -(\lambda_1+\lambda_2)P_1(t)$$

$$\lim_{\Delta t\to 0}\frac{P_1(t+\Delta t)-P_1(t)}{\Delta t} = \frac{d\,P_1(t)}{dt} = -(\lambda_1+\lambda_2)P_1(t) \quad (6.6)$$

Similarly, we obtain

$$\frac{d\,P_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t) \quad (6.7)$$

$$\frac{d\,P_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1 P_3(t) \quad (6.8)$$

$$P_1 + P_2 + P_3 + P_4 = 1$$

Two-component redundant system:

From Eq. 6.6: $\frac{d\,P_1(t)}{P_1(t)} = -(\lambda_1+\lambda_2)dt$

Integrating both sides: $\ln P_1(t) = -(\lambda_1+\lambda_2)t$

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t} \quad (6.9)$$

From Eq. 6.7: $\frac{d\,P_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2 P_2(t)$

With $e^{+\lambda_2 t}$ as an integrating factor

$$P_2(t)e^{+\lambda_2 t} = +\lambda_1\int e^{-(\lambda_1+\lambda_2)t}e^{-\lambda_2 t}dt - C$$

Thus $P_2(t) = -e^{-(\lambda_1+\lambda_2)t} + ce^{-\lambda_2 t}$ (6.10)

$P_1(0) = 1$, $P_2(0) = 0$, $P_3(0) = 0$. Thus, $c = 1$.

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t} \quad (6.9)$$

$$P_2(t) = e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t} \quad (6.10)$$

$$P_3(t) = e^{-\lambda_1 t} - e^{-(\lambda_1+\lambda_2)t} \quad (6.11)$$

$$P_4(t) = 1 - P_1(t) - P_2(t) - P_3(t) \quad (6.12)$$

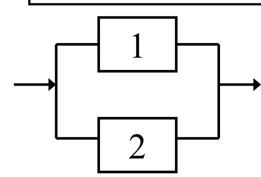
For series system:

$$R_{series} = P_1(t) = e^{-(\lambda_1+\lambda_2)t}$$

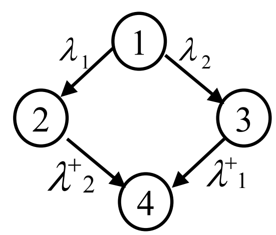
For parallel system:

$$R_{parallel} = P_1(t) + P_2(t) + P_3(t)$$
$$= e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}$$

<u>6.2 Load-Sharing System</u>



If one component doesn't work property or fail, the other component has to take *more* load.



$$\frac{d\,P_1(t)}{dt} = -(\lambda_1+\lambda_2)P_1(t) \quad (6.13)$$

$$\frac{d\,P_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2^+ P_2(t) \quad (6.14)$$

$$\frac{d\,P_3(t)}{dt} = \lambda_2 P_1(t) - \lambda_1^+ P_3(t) \quad (6.15)$$

Load sharing system:

$$\frac{d\,P_2(t)}{dt} = \lambda_1 e^{-(\lambda_1+\lambda_2)t} - \lambda_2^+ P_2(t)$$

With $e^{+\lambda_2 t}$ as an integrating factor:

$$e^{\lambda_2^+ t} P_2(t) = \int \lambda_1 e^{-(\lambda_1+\lambda_2)t} e^{\lambda_2^+ t}\,dt + C$$

$$P_2(t) = e^{-\lambda_2^+ t}\left\{\int \lambda_1 e^{-[\lambda_2^+ - (\lambda_1+\lambda_2)]t}\,dt + C\right\}$$

$$= \frac{\lambda_1 e^{-(\lambda_1+\lambda_2)t}}{\lambda_2^+ - (\lambda_1+\lambda_2)} + c e^{-\lambda_2^+ t}$$

$$c = \frac{-\lambda_1}{\lambda_2^+ - (\lambda_1+\lambda_2)} + e^{-\lambda_2^+ t}$$: Since $P_2(0) = 0$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_2^+} + \left[e^{-\lambda_2^+ t} - e^{-(\lambda_1+\lambda_2)t}\right]$$

$$P_1(t) = e^{-(\lambda_1+\lambda_2)t} \quad (6.16)$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2-\lambda_2^+}\left[e^{-\lambda_2^+ t} - e^{-(\lambda_1+\lambda_2)t}\right] \quad (6.17)$$

$$P_3(t) = \frac{\lambda_2}{\lambda_1+\lambda_2-\lambda_1^+}\left[e^{-\lambda_1^+ t} - e^{-(\lambda_1+\lambda_2)t}\right] \quad (6.18)$$

$$R(t) = P_1(t) + P_2(t) + P_3(t)$$

If we let $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_1^+ = \lambda_2^+ = \lambda^+$, then

$$R(t) = e^{-2\cdot\lambda\cdot t} \cdot \frac{2\lambda}{2\lambda-\lambda^+}\left[e^{-\lambda^+\cdot t} - e^{-2\cdot\lambda\cdot t}\right] \quad (6.19)$$

$$MTTF = \int_0^\infty R(t)\cdot dt = \frac{1}{2\lambda} + \frac{2\lambda}{2\lambda-\lambda^+}\left[\frac{1}{\lambda^+} - \frac{1}{2\lambda}\right] \quad (6.20)$$

Ex 6.1 Two generators provide electrical power. Having $\lambda = 0.01$ failure per day and $\lambda^+ = 0.10$ failure per day. Find R(10) and MTTF.

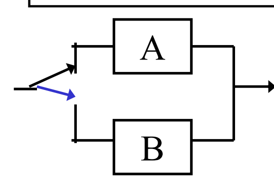
Load sharing system:

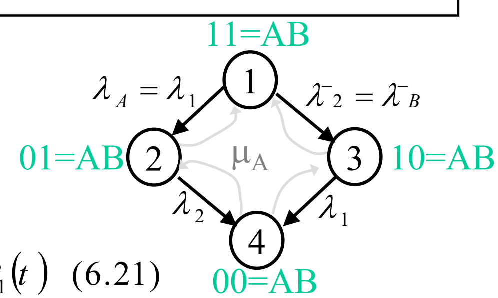
$$R(t) = e^{-2(0.01)t}\,\frac{2(0.01)}{2(0.01)-0.10^+}\left[e^{-(0.01)t} - e^{-2(0.01)t}\right]$$

$$and \quad R(10) = e^{-0.2}\,\frac{0.02}{-0.08}\left[e^{-1} - e^{-0.2}\right] = 0.9314$$

$$MTTF = \frac{1}{0.02} + \frac{0.02}{-0.08}\left[\frac{1}{0.10} - \frac{1}{0.02}\right] = 60 \quad days$$

6.3 Standby Systems





$$\frac{d\,P_1(t)}{dt} = -(\lambda_1 + \lambda^-_2)P_1(t) \quad (6.21)$$

$$\frac{d\,P_2(t)}{dt} = \lambda_1 P_1(t) - \lambda_2 P_2(t) \quad (6.22)$$

$$\frac{d\,P_3(t)}{dt} = \lambda^-_2 P_1(t) - \lambda_1 P_3(t) \quad (6.23)$$

$$P_1(t) = e^{-(\lambda_1+\lambda_2^-)t} \quad (6.24)$$

$$P_2(t) = \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}\left[e^{-\lambda_2 \cdot t} - e^{-(\lambda_1+\lambda_2^-)t}\right] \quad (6.25)$$

$$P_3(t) = e^{-\lambda_1 \cdot t} - e^{-(\lambda_1+\lambda_2^-)t} \quad (6.26)$$

$$R(t) = P_1(t) + P_2(t) + P_3(t)^{-(\lambda_1+\lambda_2^-)t}$$

$$= e^{-\lambda_1 \cdot t} + \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}\left[e^{-\lambda_2 \cdot t} - e^{-(\lambda_1+\lambda_2^-)t}\right] \quad (6.27)$$

$$MTTF = \frac{1}{\lambda_1} + \frac{\lambda_1}{\lambda_1+\lambda_2^- - \lambda_2}\left[\frac{1}{\lambda_2} - \frac{1}{\lambda_1+\lambda_2^-}\right]$$

$$= \frac{1}{\lambda_1} + \frac{1}{\lambda_2 \cdot (\lambda_1+\lambda_2^-)} \quad (6.28)$$

If there are no failures of the standby unit, $\lambda_2^- = 0$ in Eqs (6.27),(6.28). If $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_2^- = \lambda^-$, then

$$R(t) = e^{-\lambda \cdot t} + \frac{\lambda}{\lambda^-}\left[e^{-\lambda \cdot t} - e^{-(\lambda+\lambda^-)t}\right] \quad (6.29)$$

and
$$MTTF = \frac{1}{\lambda} + \frac{\lambda}{\lambda^-}\left[\frac{1}{\lambda} - \frac{1}{\lambda+\lambda^-}\right]$$

$$= \frac{1}{\lambda} + \frac{\lambda}{\lambda^-} - \frac{\lambda}{(\lambda+\lambda^-)\cdot \lambda^-}$$

$$= \frac{1}{\lambda} + \frac{1}{\lambda+\lambda^-} \quad (6.30)$$

Ex 6.2 An active generator has a failure rate of 0.01 per day. An older standby generator has a failure rate of 0.001 per day while in standby and 0.10 per day while on-line. Find R(30 days) and system MTTF.

Standby system:

$$R(t) = e^{-0.01t} + \frac{0.01}{0.01+0.001-0.1}\left[e^{-0.1t} - e^{-0.011t}\right]$$

R(30) = 0.741 – 0.11236[0.049708 – 0.7189] = 0.8162

$$MTTF = \frac{1}{0.01} + \frac{0.01}{0.1(0.01+0.001)} = 109.09 \text{ days}$$

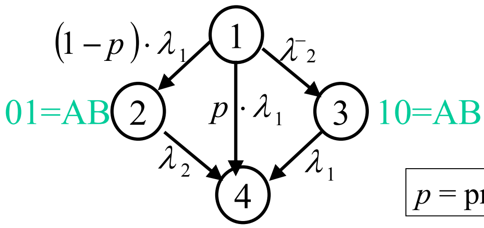
Ex 6.3 A two-component standby system with identical $\lambda = 0.002$ failure per hour and $\lambda^- = 0.0001$ failure per hour. Find the design life time on a system reliability = 0.95.

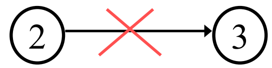
$$0.95 = R(t) = e^{-0.002t} + \frac{0.002}{0.0001}\left[e^{-0.002t} - e^{-0.0021t}\right]$$

R(173) = 0.95

Thus t_R = 173 hours

Standby System with Switching Failures





p = prob of failure of the switch

$$\frac{d\,P_1(t)}{dt} = -\left[(1-p)\lambda_1 + p\lambda_1 + \bar{\lambda}_2\right]\cdot P_1(t) = -\left(\lambda_1 + \bar{\lambda}_2\right)\cdot P_1(t)$$

$$\frac{d\,P_2(t)}{dt} = (1-p)\lambda_1 P_1(t) - \lambda_2 P_2(t)$$

$$\frac{d\,P_3(t)}{dt} = \bar{\lambda}_2 P_1(t) - \lambda_1 P_3(t) \quad \text{same as Eq.(6.23)}$$

Thus, $P_1(t) = e^{-(\lambda_1+\bar{\lambda}_2)t}$, same as Eq.(6.24)

$$P_2(t) = \frac{(1-p)\lambda_1}{\lambda_1 + \bar{\lambda}_2 - \lambda_2}\cdot\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}\right] \quad (6.32)$$

$$P_3(t) = e^{-\lambda_1 \cdot t} - e^{-(\lambda_1+\bar{\lambda}_2)t} \quad \text{, same as Eq.(6.26)}$$

$$R(t) = e^{-\lambda_1 t} + \frac{(1-p)\lambda_1}{\lambda_1 + \bar{\lambda}_2 - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\bar{\lambda}_2)t}\right] \quad (6.33)$$

If $p = 1$, the standby system has no effect and the system reliability is that of the primary unit only.

If $p = 0$, we have perfect switch, or the switch is transparent.

Ex 6.5 Consider the standby system discussed in Ex 6.2. If the prob of failure of the switch is 0.1 or 10 %, find the system reliability for a 30-day usage.

$$\text{since} \quad R(t) = e^{-\lambda_1 t} + \frac{(1-p)\lambda_1}{\lambda_1 + \bar{\lambda}_2 - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1+\bar{\lambda}_2)t}\right]$$

$$R(30) = 0.741 + \frac{(0.90)(0.01)}{0.01 + 0.001 + 0.1}\left[0.04978 - 0.7189\right]$$

$$= 0.8087$$

Compare with R(30) = 0.8162 from Ex 6.2 (with perfect switch)